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Ghost story. II. The midpoint ghost vertex

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ABSTRACT: We construct the ghost number 9 three strings vertex for OSFT in the natural normal ordering. We find two versions, one with a ghost insertion at $z = i$ and a twist-conjugate one with insertion at $z = -i$. For this reason we call them midpoint vertices. We show that the relevant Neumann matrices commute among themselves and with the matrix G representing the operator K_1. We analyze the spectrum of the latter and find that beside a continuous spectrum there is a (so far ignored) discrete one. We are able to write spectral formulas for all the Neumann matrices involved and clarify the important role of the integration contour over the continuous spectrum. We then pass to examine the (ghost) wedge states. We compute the discrete and continuous eigenvalues of the corresponding Neumann matrices and show that they satisfy the appropriate recursion relations. Using these results we show that the formulas for our vertices correctly define the star product in that, starting from the data of two ghost number 0 wedge states, they allow us to reconstruct a ghost number 3 state which is the expected wedge state with the ghost insertion at the midpoint, according to the star recursion relation.

KEYWORDS: String Field Theory, Tachyon Condensation, BRST Symmetry

Contents

1 Introduction

Wedge states are associated to an integer n and are defined in the abstract by the $*$–multiplication rule

$$|n\rangle \star |m\rangle = |n + m - 1\rangle \tag{1.1}$$

They may have different 'embodiments', [1, 2]. They are surface states and, as such, may be realized as squeezed states in the oscillator formalism or as exponentials of the operator $\mathcal{L}_0 + \mathcal{L}_0^\dagger$ applied to the vacuum; other representations are also possible. Our purpose, in the series of papers started with [3], is to find the correspondence between *the different realizations of the ghost part* of the wedge states.

We recall that in [3] we were concerned with proving the equation

$$e^{-\frac{n-2}{2}\left(\mathcal{L}_0^{(g)} + \mathcal{L}_0^{(g)\dagger}\right)}|0\rangle = \mathcal{N}_n\, e^{c^\dagger S_n b^\dagger}|0\rangle = |n\rangle \tag{1.2}$$

where $|n\rangle$ are the ghost wedge states in the oscillator formalism, which is a crucial ingredient of the analytic solution of SFT found in [4] (see also [5–26] and [27] for an updating on recent progress). It is known that the l.h.s. of this equation can also be written as a squeezed state whose defining matrix is that of a surface state (with ghost insertion at 0 in the UHP). In [3] (also referred to henceforth as I) we dealt mostly with it from the oscillator point of view. We showed that it can be cast into the midterm form in (1.2) and we diagonalized the matrix S_n in a *continuous basis of eigenvectors*. Then we proved that, *if* we are allowed to star-multiply the squeezed states representing the ghost wedge states $|n\rangle$ the same way we do for the matter wedge states and diagonalize the corresponding matrices, the eigenvalue we obtain satisfies the wedge states recursion relation. This was based on the expectation that all the (twisted) Neumann matrices entering the game could be diagonalized in the same basis (this is what happens for the matrices of the matter sector). In the course of the continuation of this research we realized that the expectation of I was a bit too optimistic and had to readjust our line of arguments. The main reason for this is that the spectral theory of the Neumann matrices that characterize the ghost sector of the three strings vertex and wedge states is significantly different from ordinary spectral theory of real symmetric matrices, which are the basic example of Neumann matrices in the matter sector. Once the eigenvalues and eigenvectors of the latter matrices are given,

their reconstruction via the spectral formula is unique. In the ghost case instead the spectral formula is not uniquely determined but depends on the integration contour over the continuous spectrum: this is one of the basic results of our analysis. It should be clarified that such spectral formulas we obtain for the ghost Neumann matrices are not derived on the basis of general theorems in operator theory, which to our best knowledge do not exist in the literature, but on a heuristic basis. We thinks we have checked them beyond any doubt both numerically and using consistency with other methods.

In this paper we introduce a three-strings vertex for the ghost part in order to be able to explicitly perform the star product in (1.1), up to a midpoint ghost insertion. Moreover we complete the spectral analysis of the ghost bases by computing the relevant discrete bases of eigenvectors (which were missing in [3, 28]). Finally we show that the states in the l.h.s. of (1.2) do satisfy the recursion relations for the wedge states (the r.h.s.), although not in the form expected in I. We show in fact that only the eigenvalues of the relevant matrices satisfy the appropriate recursion relations. Based on this, we can reconstruct, in the sense mentioned above, Neumann matrices which represent ghost number 3 states and show that the latter are surface states with a midpoint insertion, representing, at $gh=3$, the expected wedge states. So, it is true that in the ghost number 3 sector things work much as in the matter sector. However the same is not true for the ghost number 0 sector. In fact what remains to be done is reconstructing from the ghost number 3 the ghost number 0 wedge states we started from (eq. (1.2)). This would close the circle and fully justify our claim about the consistency of our three strings ghost vertex and the correctness of (1.2). This rather non-trivial task will be carried out in another paper [29], referred to as III.

Notation. Any infinite matrix we meet in this paper is either square short or long legged, or lame. In this regard we will often use a compact notation: a subscript $_s$ will represent an integer label n running from 2 to ∞, while a subscript $_l$ will represent a label running from -1 to $+\infty$. So Y_{ss}, Y_{ll} will denote square short and long legged matrices, respectively; Y_{sl}, Y_{ls} will denote short-long and long-short lame matrices, respectively. With the same meaning we will say that a matrix is $(ll), (ss), (sl)$ or (ls). The (ss) part of a matrix M will be referred to as the bulk of M. In a similar way we will denote by V_s and V_l a short and long infinite vector, to which the above matrices naturally apply. Moreover, while n, m represent generic matrix indices, at times we will use N, M to represent 'long' indices, i.e. $N, M \geq -1$. In this case n, m will represent short indices, i.e. $n, m \geq 2$.

We will also use the symbol $\mathcal{C}$ to represent the twist matrix, $\mathcal{C}_{n,m} = (-1)^n \delta_{n,m}$. Given any matrix M, we generically represent the twisted matrix $\mathcal{C}M$ by $\tilde{M}$. Finally we use the symbol gh to denote the ghost number.

1.1 A summary of the results

Since the paper is rather long and elaborate we would like to start with an outline of it and a summary of the main results.

To start with we first recall the basic anti-commutator for the b, c ghost oscillators and bpz transformation properties

$$[c_n, b_m]_+ = \delta_{n+m,0}, \qquad bpz(c_n) = -(-1)^n c_{-n}, \qquad bpz(b_n) = (-1)^n b_{-n}, \qquad bpz(|0\rangle) = \langle 0|$$

where $|0\rangle$ is the SL(2,R)–invariant vacuum. Next we define the state $|\hat{0}\rangle = c_{-1}c_0c_1|0\rangle$ and the tensor product of states

$$_{123}\langle\hat{\omega}| = {}_1\langle\hat{0}|_2\langle\hat{0}|_3\langle\hat{0}| \tag{1.3}$$

carrying total $gh = 9$, and

$$|\omega\rangle_{123} = |0\rangle_1|0\rangle_2|0\rangle_3 \tag{1.4}$$

carrying total $gh=0$. They satisfy $_{123}\langle\hat{\omega}|\omega\rangle_{123} = 1$. Finally we write down the general form of the three strings vertices we will find below (section 2). The first two are

$$\langle\hat{V}_{(\pm i)3}| = \hat{\mathcal{K}}_{(\pm i)\ 123}\langle\hat{\omega}|e^{\hat{E}_{(\pm i)}}, \qquad \hat{E}_{(\pm i)} = -\sum_{r,s=1}^{3}\sum_{n,m}^{\infty}c_n^{(r)}\hat{V}_{(\pm i)nm}^{rs}b_m^{(s)} \tag{1.5}$$

where

$$\hat{V}_{(i)nm}^{rs} = \oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\frac{1}{z^{n-1}}\frac{1}{w^{m+2}}\left(\frac{\left(\frac{d}{dz}f_r(z)\right)^2}{\frac{d}{dw}f_s(w)}\frac{1}{f_r(z)-f_s(w)}\left(\frac{f_s(w)}{f_r(z)}\right)^3 - \frac{\delta^{rs}}{z-w}\right) \tag{1.6}$$

and

$$\hat{V}_{(-i)nm}^{rs} = \oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\frac{1}{z^{n-1}}\frac{1}{w^{m+2}}\left(\frac{\left(\frac{d}{dz}f_r(z)\right)^2}{\frac{d}{dw}f_s(w)}\frac{1}{f_r(z)-f_s(w)} - \frac{\delta^{rs}}{z-w}\right) \tag{1.7}$$

The labels $(\pm i)$ refer to the ghost insertion at the string midpoint i and image point $-i$, respectively (see below). These Neumann matrices are complex.

We will also use a third auxiliary vertex (a sort of average of the previous two) whose Neumann matrices are real

$$\langle\hat{V}_3| = \hat{\mathcal{K}}_{123}\langle\hat{\omega}|e^{\hat{E}}, \qquad \hat{E} = -\sum_{r,s=1}^{3}\sum_{n,m}^{\infty}c_n^{(r)}\hat{V}_{nm}^{rs}b_m^{(s)} \tag{1.8}$$

where

$$\hat{V}_{nm}^{rs} = \frac{1}{2}\oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\frac{1}{z^{n-1}}\frac{1}{w^{m+2}}\left(\frac{\left(\frac{d}{dz}\ln f_r(z)\right)^2}{\frac{d}{dw}\ln f_s(w)}\frac{f_r(z)+f_s(w)}{f_r(z)-f_s(w)} - \frac{\delta^{rs}}{z-w}\right) \tag{1.9}$$

All these vertices satisfy cyclicity

$$\hat{V}_{nm}^{rs} = \hat{V}_{nm}^{r+1,s+1}, \qquad \hat{V}_{(\pm i)nm}^{rs} = \hat{V}_{(\pm i)nm}^{r+1,s+1} \tag{1.10}$$

The third vertex satisfies twist-covariance

$$\hat{V}_{nm}^{rs} = (-1)^{n+m}\hat{V}_{nm}^{sr} \tag{1.11}$$

while the first two are twist conjugate

$$\hat{V}_{(-i)nm}^{rs} = (-1)^{n+m}\hat{V}_{(i)nm}^{sr} \tag{1.12}$$

The latter are BRST invariant. Dual vertices can also be defined. We will show that the twisted Neumann matrices of each vertex commute.

The constants $\mathcal{K}$ and $\mathcal{K}_{(\pm i)}$ turn out to be 1.

In the previous formulas

$$f_r(z_r) = \alpha^{2-r} f(z_r) \,, \; r = 1, 2, 3 \tag{1.13}$$

where

$$f(z) = \Big(\frac{1 + iz}{1 - iz}\Big)^{\frac{2}{3}} \tag{1.14}$$

Here $\alpha = e^{\frac{2\pi i}{3}}$.

In section 3 we will show that the twisted Neumann matrices of all the vertices just introduced commute with the matrix G, which represents the operator $K_1 = L_1 + L_1^\dagger$. This allows us to diagonalize the matrices that commute with G on the basis of its eigenvectors. In section 4 we explicitly compute the bases corresponding to the discrete spectrum of G, while the continuous spectrum had already been computed in [30–32]. We also write down the spectral formula for G and notice that it depends on the contour one takes in order to integrate over the continuous eigenvalue κ: only in a certain range of $\Im(\kappa)$ do we correctly reproduce G. We also give (partial) reconstruction formulas for the matrices A, B, C, D of I.

In section 5 we write down spectral formulas for the (twisted) Neumann matrices of the above constructed vertices. We show that the integration contour over the continuous spectrum plays a fundamental role. In fact different vertices have the same spectral formulas but differ by the integration contour and can be obtained from one another by changing it.

The main purpose of section 6 is to extract information about the eigenvalues of the Neumann coefficients of the ghost number 0 wedge states from solving the KP equation, [48], as was done in I. The main difference with I is that we do not use commutativity of the matrices A, B, C, D but solve the equation for their eigenvalues. In such a way we are able to prove that both the continuous and discrete eigenvalues of the wedge states satisfy the appropriate recursion relation. With these results at hand, in section 7 we pass to the task of reconstructing the twisted Neumann matrices of the ghost number 3 wedge states. Once again the integration path over the continuous spectrum plays a crucial role and allows us to pass from one possible representation to another of these states. It is clear that in so doing we are assuming that the eigenvalues are common to all the representations of a given wedge state both with ghost number 3 and with ghost number 0. This assumption turns out to be correct but will be fully justified only in paper III.

Finally, five appendices contains auxiliary material, calculations and complements.

2   The three strings vertex

In order to construct the ghost three string vertex in the oscillator formalism (for previous literature, see [33–37]; problems related to the present paper are treated in [38–45]) we

have to face a number of problems which are not met in the matter sector. The first is normal ordering. Let us recall that one can envisage two main types of normal orderings, which we have called in [3] the *natural* and *conventional normal ordering*. The former is the obvious normal ordering required when the vacuum is $|0\rangle$, the latter is instead requested by the vacuum state $c_1|0\rangle$ (of course, in principle, one could consider other possibilities). A second problem is generated by the ghost insertions, which are a priori free. It is clear that the three strings vertex will depend to some extent both on the normal ordering and the ghost insertion. Finally the vertices must be BRST invariant.

To start with, in this paper we will use the natural normal ordering. This is at variance with the existing ghost three strings vertex [33–37], which is based on the conventional normal ordering. This innovation is required by the new non–perturbative analytic solution of SFT found by Schnabl, [4], where ghost number 0 wedge states are used, for which the old ghost vertex is ineffective, see for instance [28].

Using the definitions (1.5), (1.8), our aim now is to explicitly compute $\hat{V}^{rs}_{(\pm i)nm}, \hat{V}^{rs}_{nm}$. The method is well–known: one expresses the propagator $\ll c(z)b(w) \gg$ (see appendix A) in two different ways, first as a CFT correlator and then in terms of $\hat{V}_3$ and equates the two expressions after mapping them to the disk via the maps (1.13). However this recipe leaves several uncertainties.

First we have to insert the three c zero modes. We can either, for instance, insert three separate fields $c(z_i)$, (A.1), or use $Y(z) = \frac{1}{2}\partial^2 c(z)\partial c(z)c(z)$. In order to pair ghost number 3 and ghost number 0 states so that they preserve their conformal properties, we should use a ghost number 3 primary field insertion with vanishing conformal weight. This implies the use of Y, which has this property. Even so there remain many possibilities. Let us make the obvious remark that, given the vacuum $|0>$, there are many ways to define a conjugate vacuum $|0^c>$ such that $<0^c|0>=<0|0^c>=1$. The simplest example is given by $|0^c>=Y(0)|0>=c_{-1}c_0c_1|0>$. However this is not the only possible choice since $\partial_z <0|Y(z)|0>=0$. So, in principle, any choice of $|0^c>=Y(z)|0>$ is a good conjugate vacuum and we can choose the insertion point as we like.

The above can be understood in terms of Q_B cohomology. Remembering that

$$\{Q, c(z)\} = c\partial c(z),$$

we have

$$\{Q, Y(z)\} = 0, \qquad \text{and} \qquad \partial Y(z) = Q(\ldots).$$

This means that the point where one inserts Y is irrelevant when the other string fields in the game are in the kernel of Q. This is in particular true for surface states in critical dimension. For any surface state Σ we have

$$\partial_z \langle\Sigma|Y(z)|\Sigma\rangle = 0$$

if $Q|\Sigma\rangle = 0$.

In defining the vertex, however, the place where Y is inserted matters. This is because the $*$–product treats the midpoint as special (it is the only point which is common to the three interacting strings). So, out of the infinite places where we could insert Y, we

make the most symmetric choice of inserting Y at the midpoint. Since Y is a weight zero primary, this will not cause the typical divergences of midpoint insertions. The vertex we are constructing is thus meant to perform the $*$–product (which is ghost number preserving) and then to add a Y midpoint insertion to the result. Calling $\langle \hat{V}_3 |$ such a vertex, we can define it symbolically as

$$\langle \hat{V}_3 | | \psi_1 \rangle | \psi_2 \rangle = \langle \psi_1 * \psi_2 | \Big(Y(i) \,, Y(-i) \Big) \tag{2.1}$$

In other words, calling $\langle V_3 |$ the usual three–strings vertex without insertions, we can write

$$\langle \hat{V}_3 | = \langle V_3 | \Big(Y(i) \,, Y(-i) \Big) = \Big(\langle \hat{V}_{(i)} | \,, \langle \hat{V}_{(-i)} | \Big) \tag{2.2}$$

meaning that, as we will see, we need both insertions in order to correctly represent the star product (this is just the doubling trick).

In the natural normal ordering it is impossible to represent $\langle V_3 |$ in a squeezed state form which is cyclic in the string indices. That is not a problem, in principle, but it would give rise to very complicated Neumann coefficients matrices. On the other hand the midpoint inserted vertex $\langle \hat{V}_3 |$ is expressed in terms of two cyclic squeezed states: each of them can actually be used independently of each another. Computations with $\langle \hat{V}_{(i)} |$ will be related to $\langle \hat{V}_{(-i)} |$ by twist–conjugation. The price we have to pay for this choice in the vertex is that, when we midpoint–multiply 2 $gh = 0$ states, we get a $gh = 3$ result. Going back to $gh = 0$ will be the subject of III.

2.1 Three zero modes insertion

We start by inserting the operator Y at the point t (for simplicity we understand the dependence on t in the vertex, until further notice). We use the correlator (A.1) in appendix A and compare

$$\langle f_j \circ Y(t) \, f_r \circ c^{(r)}(z) \, f_s \circ b^{(s)}(w) \rangle \tag{2.3}$$

with

$$\langle \hat{V}_3 | R(c^{(r)}(z) \, b^{(s)}(w)) | \omega \rangle_{123} \tag{2.4}$$

where R denotes radial ordering. If $::$ denotes the natural normal ordering, we have for instance (see appendix A)

$$R(c(z) \, b(w)) = \sum_{n,k} : c_n \, b_k : \, z^{-n+1} w^{-k-2} + \frac{1}{z - w} \tag{2.5}$$

This should be inserted inside (2.4). Let us refer to the last term in (2.5) as the *ordering term*.

We first compute the $\hat{\mathcal{K}}$ constant. By making use of $\langle 0 | Y(t) | 0 \rangle = 1$ for any t, we have

$$\langle \hat{V}_3 | \omega \rangle_{123} = \hat{\mathcal{K}} = \langle f_j \circ Y(t) \rangle = 1 \tag{2.6}$$

for any j. Now

$$\langle \hat{V}_3 | R(c^{(r)}(z)\, b^{(s)}(w)) | \omega \rangle_{123} = \langle \hat{V}_3 | \sum_{n,k} : c_n^{(r)}\, b_k^{(s)} : \, z^{-n+1} w^{-k-2} + \frac{\delta^{rs}}{z-w} | \omega \rangle_{123}$$

$$= -\hat{V}_{kn}^{sr}\, z^{n+1} w^{k-2} + \frac{\delta^{rs}}{z-w} \tag{2.7}$$

On the other hand, from direct computation,

$$\langle f_r \circ c(z)\, f_s \circ b(w)\, f_j \circ Y(t) \rangle = \frac{(f_s'(w))^2}{f_r'(z)} \frac{1}{f_r(z) - f_s(w)} \left(\frac{f_j(t) - f_r(z)}{f_j(t) - f_s(w)} \right)^3 \tag{2.8}$$

Comparing the last two equations and using (2.6) we get

$$\hat{V}_{kn}^{sr} = -\oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^{n+2}} \frac{1}{w^{k-1}} \cdot \left(\frac{(f_s'(w))^2}{f_r'(z)} \frac{1}{f_r(z) - f_s(w)} \left(\frac{f_j(t) - f_r(z)}{f_j(t) - f_s(w)} \right)^3 - \frac{\delta^{rs}}{z-w} \right) \tag{2.9}$$

After obvious changes of indices and variables we end up with

$$\hat{V}_{nm}^{rs} = \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^{n-1}} \frac{1}{w^{m+2}} \cdot \left(\frac{(f_r'(z))^2}{(f_s'(w))} \frac{1}{f_r(z) - f_s(w)} \left(\frac{f_s(w) - f_j(t)}{f_r(z) - f_j(t)} \right)^3 - \frac{\delta^{rs}}{z-w} \right) \tag{2.10}$$

After some elementary algebra, using $f'(z) = \frac{4i}{3} \frac{1}{1+z^2} f(z)$, one finds

$$\hat{V}_{nm}^{rs} = \frac{1}{3}(E_{nm} + \bar{\alpha}^{r-s} U_{nm} + \alpha^{r-s} \bar{U}_{nm}) \tag{2.11}$$

where

$$E_{nm} = \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^{n+1}} \frac{1}{w^{m+1}} \left[\left(\frac{1}{1+zw} - \frac{w}{w-z} \right) \cdot (1 - p_t(z,w)) - \frac{z^2}{w} \frac{1}{z-w} \right] \tag{2.12}$$

$$U_{nm} = \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^{n+1}} \frac{1}{w^{m+1}} \left[\frac{f(z)}{f(w)} \left(\frac{1}{1+zw} - \frac{w}{w-z} \right) \cdot (1 - p_t(z,w)) - \frac{z^2}{w} \frac{1}{z-w} \right] \tag{2.13}$$

$$\bar{U}_{nm} = \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^{n+1}} \frac{1}{w^{m+1}} \left[\frac{f(w)}{f(z)} \left(\frac{1}{1+zw} - \frac{w}{w-z} \right) \cdot (1 - p_t(z,w)) - \frac{z^2}{w} \frac{1}{z-w} \right] \tag{2.14}$$

for $t = \pm i$. In the above equations

$$p_t(z,w) = \frac{t(w-z)(1+wz)}{w(t-z)(1+tz)} \tag{2.15}$$

This function enjoys the properties

$$p_t\left(-\frac{1}{z}, w\right) = p_t(z,w), \quad p_t\left(z, -\frac{1}{w}\right) = p_t(z,w), \quad p_t(z,z) = 0, \quad p_t(0,z) = 1 \tag{2.16}$$

which will be of great importance later on.

It is immediate to check cyclicity (with $t = \pm i$)

$$\hat{V}_{nm}^{rs} = \hat{V}_{nm}^{r+1,s+1},$$

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Moreover we have the twist covariance property

$$\hat{V}^{rs}_{(i)nm} = (-1)^{n+m}\hat{V}^{sr}_{(-i)nm}$$

that is the vertex with Y insertion at i is twist conjugate to the one with insertion at $-i$. This is due, in particular, to the property

$$p_i(-z,-w) = p_{-i}(z,w) \tag{2.17}$$

So we have a couple of twist–conjugate vertices. Due to the considerations at the beginning of this section, these two vertices are BRST invariant. As we will see in the sequel, they have the properties we need, therefore we stick to them even though they are complex. They are the two vertices defined by formulas (1.6), (1.7). The corresponding $E, U, \bar{U}$ are the ones defined by eqs. (2.12), (2.13), (2.14), with $t = i$ and $-i$, respectively, in p_t.

2.1.1 The midpoint Neumann coefficients

In conclusion, our midpoint vertices are defined as

$$\hat{V}^{rs}_{(\pm i)nm} = \frac{1}{3}(E_{(\pm i)nm} + \bar{\alpha}^{r-s}U_{(\pm i)nm} + \alpha^{r-s}\bar{U}_{(\pm i)nm}) \tag{2.18}$$

in terms of the quantities

$$E_{(\pm i)} = \mathcal{E}_{(\pm i)} + Z, \qquad U_{(\pm i)} = \mathcal{U}_{(\pm i)} + Z, \qquad \bar{U}_{(\pm i)} = \bar{\mathcal{U}}_{(\pm i)} + Z \tag{2.19}$$

where

$$\mathcal{E}_{(\pm i)nm} = \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^{n+1}} \frac{1}{w^{m+1}} \left(\frac{1}{1+zw} - \frac{w}{w-z}\right)(1 - p_{\pm i}(z,w)) \tag{2.20}$$

$$\mathcal{U}_{(\pm i)nm} = \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^{n+1}} \frac{1}{w^{m+1}} \frac{f(z)}{f(w)} \left(\frac{1}{1+zw} - \frac{w}{w-z}\right)(1 - p_{\pm i}(z,w)) \tag{2.21}$$

$$\bar{\mathcal{U}}_{(\pm i)nm} = \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^{n+1}} \frac{1}{w^{m+1}} \frac{f(w)}{f(z)} \left(\frac{1}{1+zw} - \frac{w}{w-z}\right)(1 - p_{\pm i}(z,w)) \tag{2.22}$$

with the ordering term

$$Z_{nm} = \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^{n+1}} \frac{1}{w^{m+1}} \left(-\frac{z^2}{w}\frac{1}{z-w}\right) \tag{2.23}$$

2.2 The average (real) vertex

In addition to these two vertices we will construct a third one which is twist invariant and real, although it does not evidently respect BRST invariance.

One way to get a twist invariance vertex is to average between the two above, that is to make the replacement

$$\left(\frac{f_s(w) - f_j(t)}{f_r(z) - f_j(t)}\right)^3 \longrightarrow \frac{1}{2}\left(\left(\frac{f_s(w) - f_j(t)}{f_r(z) - f_j(t)}\right)^3 + \left(\frac{f_s(w) - f_j(\bar{t})}{f_r(z) - f_j(\bar{t})}\right)^3\right) \tag{2.24}$$

in the above definitions, mimicking the method of images. We stress that here we refer to the average of the vertex exponents.

This leads to (2.28), (2.12), (2.13), (2.14) with p_t replaced by

$$p_0(z,w) \equiv \frac{1}{2}(p_i(z,w) + p_{-i}(z,w)) = \frac{(z-w)(1+zw)(z^2-1)}{w(1+z^2)^2} \tag{2.25}$$

However this choice produces a singularity in the product $\mathcal{U}^2$ (see subsection 2.4), a singularity which is due to the double pole of $p_0(z,w)$ at $z=i$ and $z=-i$. The definition of the twist–invariant midpoint vertex requires a not a priori obvious modification, which is as follows. We replace p_t with p_0 in $\mathcal{E}$, with p_i in $\mathcal{U}$ and with p_{-i} in $\bar{\mathcal{U}}$, (2.12), (2.13), (2.14) respectively. We notice that, beside the properties (2.16), one has

$$p_0(-z,-w) = p_0(z,w) \tag{2.26}$$

As is easily verified, this property guarantees twist–invariance of the Neumann matrices. We will denote the corresponding Neumann matrices simply by $\hat{V}^{rs}_{nm}$.

In summary the average (regularized) vertex is defined in terms of $U_{(i)}, \bar{U}_{(-i)}$ and

$$E = \mathcal{E} + Z, \qquad \mathcal{E} = \frac{1}{2}\Big(\mathcal{E}_{(i)} + \mathcal{E}_{(-i)}\Big) \tag{2.27}$$

as follows

$$\hat{V}^{rs}_{nm} = \frac{1}{3}(E_{nm} + \bar{\alpha}^{r-s}U_{(i)nm} + \alpha^{r-s}\bar{U}_{(-i)nm}) \tag{2.28}$$

Now one can easily show that the Neumann matrices $\hat{V}^{rs}_{nm}$ can be written in the compact form (1.9) (apart from the ordering term).

2.3 Two remarks

The matrices $\hat{V}^{rs}_{nm}, \hat{V}^{rs}_{(\pm i)nm}$ are all sl. However, when $r=s$, it is always possible to add to them an upper left 3×3 matrix z, where $z_{ij} = \delta_{i+j,0}$, with $-1 \leq i,j \leq 1$. The addition of the matrix z to $\hat{V}^{rr}$ does not change the vertex provided we understand that the expression of the vertex is normal ordered, since, in the definition (1.8), the vertex is applied to the vacuum $\langle \hat{0}|$. In fact we have more:

$$\langle \hat{0}| : e^{c_i\,\tau_{ij}\,b_j + c_n V_{nM} b_M} := \langle \hat{0}|e^{c_n V_{nM} b_M}$$

for any matrix τ_{ij}. This ambiguity is allowed by the formalism and actually it turns out to be very useful. This remark will be crucial in the sequel.

Another remark concerning the just defined vertices is the following. While the expressions $\mathcal{E}, \mathcal{U}$ and $\bar{\mathcal{U}}$ are ambiguous, due to the presence of the factor $1/(z-w)$, in (2.19) any ambiguity has disappeared. This is evident for E, but is true also for U and $\bar{U}$. For instance

$$\begin{aligned}
\frac{f(z)}{f(w)}\frac{w}{z-w} &= \frac{f(w) + (z-w)f'(w) + 1/2(z-w)^2 f''(w) + \dots}{f(w)}\frac{w}{z-w} \\
&= \frac{w}{z-w} + wf'(w) + \frac{1}{2}(z-w)wf''(w) + \dots
\end{aligned}$$

Of course only the first term in the r.h.s. is ambiguous when inserted in the double contour integral (2.21), but it is cancelled by the ordering term. Therefore all the double integrals above are unambiguous. But if we evaluate separately (as it will happen) $\mathcal{U}$ and Z, for instance, we have to be careful to use the same prescriptions, because each separate term is ambiguous.

Finally we record the twist properties

$$\mathcal{C}E_{(i)} = E_{(-i)}\mathcal{C}, \qquad \mathcal{C}U_{(\pm i)} = \bar{U}_{(\mp i)}\mathcal{C} \tag{2.29}$$

2.4 Fundamental properties of the Neumann coefficients

In this subsection we will analytically prove certain fundamental relations for the matrices (2.20), (2.21), (2.22) and (2.23), following the methods of [47]. We remark that the analytic proof in this case is essential, because the numerical analysis, while confirming the analytic results, is hindered by the poor convergence properties of the product matrices.

2.4.1 $\mathcal{U}_{(p)}\mathcal{U}_{(p')}$

Our first aim is to evaluate the product $(\mathcal{U}_{(p)}\mathcal{U}_{(p')})_{nm}$ where p and p' stand for either i or $-i$ and denote generically the dependence on $p_{\pm i}$. Since this result is specially important we present the calculation in full detail as a model for many others that occur in the paper.

Let us consider the product $\sum_{k=-1}^{\infty} \mathcal{U}_{(p)nk} \mathcal{U}_{(p')km}$. In the first $\mathcal{U}$ we use the integration variables z and ζ and in the second θ and w. We assume $|z| < |\zeta|$ and $|\theta| > |w|$. This means that we have first to integrate in ζ and θ and then in z and w. This prescription is arbitrary. We have to be careful to use the same prescription when computing the other pieces of $U_{(p)}U_{(p')}$.

We use for $\mathcal{U}$ the definition above, (2.21), and perform the intermediate summation in $\sum_{k=-1}^{\infty} \mathcal{U}_{(p)nk} \mathcal{U}_{(p')km}$:

$$\sum_{k=-1}^{\infty} \frac{1}{(\zeta\theta)^{k+1}} = \frac{\zeta\theta}{\zeta\theta - 1} \tag{2.30}$$

This is true if $|\zeta\theta| > 1$. If the latter condition holds we have

$$\sum_{k=-1}^{\infty} \mathcal{U}_{(p)nk} \mathcal{U}_{(p')km} = \oint \frac{dz}{2\pi i} \frac{1}{z^{n+1}} \oint \frac{d\zeta}{2\pi i} \oint \frac{d\theta}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{w^{m+1}} \frac{\zeta\theta}{\zeta\theta - 1} \tag{2.31}$$

$$\cdot \frac{f(z)}{f(\zeta)}\Big(\frac{1}{1+z\zeta} - \frac{\zeta}{\zeta-z}\Big)\Big(1 - p(z,\zeta)\Big) \cdot$$

$$\cdot \frac{f(\theta)}{f(w)}\Big(\frac{1}{1+\theta w} - \frac{w}{w-\theta}\Big)\Big(1 - p'(\theta,w)\Big) = *$$

We notice that $p(z,\zeta)$ has a double pole in $z = i$ and a simple pole in $\zeta = 0$. $p'(\theta, w)$ has a double pole in $\theta = \pm i$ and a simple pole in $w = 0$. In order to avoid the pole at $\theta = i$ it is more convenient to integrate first with respect to ζ. In the integrand there are poles in $\zeta = z, -\frac{1}{z}, \frac{1}{\theta}$. In order to guarantee $|\zeta\theta > 1|$ we have to take $|\zeta| > \frac{1}{|\theta|}$. Therefore the integration contour in ζ will include the poles in $z, \frac{1}{\theta}$, but excludes $-\frac{1}{z}$.

So we take $|\zeta| > 1$ and $|\theta| < 1$. Notice that we have

$$|z| < |\zeta|, \qquad |\theta| > |w|, \qquad |\zeta| > \frac{1}{|\theta|}, \qquad \frac{1}{|z|} > |\zeta| > \frac{1}{|\theta|}, \quad \text{i.e.} \quad |\theta| > |z| \qquad (2.32)$$

To comply with the condition $|\zeta| > 1$ we deform the ζ contour while keeping the θ contour fixed. In doing so we have to be careful to avoid possible singularities in ζ. The latter are poles at $\zeta = z, -\frac{1}{z}, \frac{1}{\theta}$ and branch cuts at $\zeta = \pm i$, due to the $f(\zeta)$ factor. One can deform the ζ contour in such a way as to keep the pole at $-\frac{1}{z}$ external to the contour, since the z contour is as small as we wish around the origin. But, of course, one cannot avoid the branch points at $\zeta = \pm i$. To make sense of the operation we introduce a regulator $K > 1$ and modify the integrand by modifying $f(\zeta)$

$$f(\zeta) \to f_K(\zeta) = \left(\frac{K + i\zeta}{K - i\zeta} \right)^{\frac{2}{3}}$$

We will take K as large as needed and eventually move back to $K = 1$.

Under these circumstances we can safely perform the summation over k, make the replacement (2.30) in the integral and get (2.31). Now we can integrate over ζ. The integration contour only surrounds $z, \frac{1}{\theta}$. So (2.31) becomes

$$\left\{ \zeta = \frac{1}{\theta} \right\} \qquad * = \oint \frac{dz}{2\pi i} \frac{1}{z^{n+1}} \oint \frac{d\theta}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{w^{m+1}} \left[\frac{1}{\theta} \frac{f(z)}{f_K(\frac{1}{\theta})} \left(\frac{\theta}{\theta + z} - \frac{1}{1 - z\theta} \right) \cdot \right.$$

$$\cdot \left(1 - p(z, -\theta) \right) \frac{f(\theta)}{f(w)} \left(\frac{1}{1 + \theta w} - \frac{w}{w - \theta} \right) \cdot$$

$$\cdot \left(1 - p'(\theta, w) \right) \qquad (2.33)$$

$$\left\{ \zeta = z \right\} \qquad - \frac{\theta z^2}{\theta z - 1} \frac{f(\theta)}{f(w)} \left(\frac{1}{1 + \theta w} - \frac{w}{w - \theta} \right)$$

$$\left. \cdot \left(1 - p'(\theta, w) \right) \right] = **$$

where, on the left, in curly brackets we denote the pole that gives rise to the contribution in the body of the formula.

Next we wish to integrate with respect to θ. There are poles at $\theta = -z, \frac{1}{z}, w, -\frac{1}{w}$ and possibly at $\theta = \pm i$, and branch cuts starting and ending at $\theta = \pm i$ and at $\theta = \pm \frac{i}{K}$ (no poles at $\theta = 0, \infty$!). The singularities trapped within the θ contour of integration are the poles at $\theta = -z, w$. Since above we had $K > |\theta| > \frac{1}{|\zeta|}$, it follows that $|\theta| > \frac{1}{K}$. Therefore also the branch points at $\theta = \pm \frac{i}{K}$ of $f_K(1/\theta)$ are trapped inside the θ contour and we have to compute the relevant contribution to the integral. Let us call this cut $\mathfrak{c}_{1/K}$ and let us fix it to be the semicircle of radius $1/K$ at the l.h.s. of the imaginary axis; the contour that surrounds it excluding all the other singularities will be denoted $C_{1/K}$. The other cut, due to $f(\theta)$, with branch points at $\theta = \pm i$, will be denoted $\mathfrak{c}_1$; the contour that surrounds it (another semicircle of radius 1) excluding all the other singularities will be denoted C_1. The forthcoming argument requires that we split the branch point at $\theta = i$ from the pole

at the same point coming from $p'(\theta, w)$. Therefore we will introduce a regulator in $p'(\theta, w)$ to move away this singularity and return eventually to the initial condition. This regulator is simply to help keeping the branch point and the pole of $p'(\theta, w)$ at $\theta = i$ distinct. The role of the poles of $p'(\theta, w)$ at $\theta = i$ will be analyzed further on.

Evaluating (2.33) we get

$$
\begin{aligned}
\{\theta = w\} \qquad ** = &\oint \frac{dz}{2\pi i} \frac{1}{z^{n+1}} \oint \frac{dw}{2\pi i} \frac{1}{w^{m+1}} \left[-\frac{f(z)}{f_K(\frac{1}{w})} \right. \\
&\cdot \left(\frac{1}{1 - zw} - \frac{w}{z + w} \right) \left(1 - p(z, -w) \right) \\
&+ \frac{z^2 w^2}{1 - zw} \\
\{\theta = -z\} \qquad &+ \frac{f(z)}{f_K(-\frac{1}{z})} \left(\frac{1}{1 - zw} - \frac{w}{z + w} \right) \left(1 - p'(-z, w) \right) \left. \frac{f(-z)}{f(w)} \right] \\
&+ \oint_{C_{1/K}} d\theta (\dots)
\end{aligned}
\tag{2.34}
$$

where the last term refers to the integral along the contour $C_{1/K}$. We have used

$$
\frac{zw}{zw - 1} - \frac{z}{z + w} = \frac{w}{z + w} - \frac{1}{1 - zw}
$$

The problem now is to evaluate the integral around the cut. Fortunately this can be reduced to an evaluation of contributions from poles. To see this, we first recall the properties of $f(z)$. It is easy to see that

$$
f(1/z) = \gamma f(-z) \quad \text{and} \quad f(-z) = 1/f(z)
\tag{2.35}
$$

This comes from

$$
f\left(\frac{1}{z} \right) = \left(\frac{1 + \frac{i}{z}}{1 - \frac{i}{z}} \right)^{\frac{2}{3}} = \left(-\frac{1 - iz}{1 + iz} \right)^{\frac{2}{3}}
$$

Above γ is either $1, \alpha$ or $\bar{\alpha}$, depending on what Riemann sheet we choose. However, denoting with an arrow the effect of a transformation $\zeta \to -\frac{1}{\zeta}$ we get

$$
f\left(-\frac{1}{\zeta} \right) = \gamma f(-\zeta) \to \gamma f\left(\frac{1}{\zeta} \right) = \gamma^2 f(-\zeta)
$$

On the other hand

$$
f\left(\frac{1}{\zeta} \right) \to f(-\zeta)
$$

Thus $\gamma^2 = 1$, which implies $\gamma = 1$. We remark that this result comes from requiring that the entry of f takes values on a Riemann sphere. The value of γ, however, does nott really matter provided we choose always the same sheet.

Therefore, in the limit $K \to 1$, the factor $f_K(1/\theta)/f(\theta)$ tends, up to the γ factor, to $(f(-\theta))^2$. As a consequence, in the same limit, the integral of $(\ldots)$ around the $\mathfrak{c}_{1/K}$ cut is the same as the integral around the $\mathfrak{c}_1$ cut. To be more explicit in (2.34) we have

$$\oint_{C_{1/K}} \frac{d\theta}{2\pi i} \frac{f(\theta)}{f_K(\frac{1}{\theta})} \ldots = \frac{1}{\gamma} \oint_{C_{1/K}} \frac{d\theta}{2\pi i} f_K(\theta) f(\theta) \ldots$$

In this expression the relevant cut is $\mathfrak{c}_{1/K}$. On the other hand

$$\oint_{C_1} \frac{d\theta}{2\pi i} \frac{f(\theta)}{f_K(\frac{1}{\theta})} \ldots = \frac{1}{\gamma} \oint_{C_1} \frac{d\theta}{2\pi i} f_K(\theta) f(\theta) \ldots$$

In this expression the relevant cut is $\mathfrak{c}_1$. It is evident that in the limit $K \to 1$ the two expressions become one and the same.

At this point it is convenient to take, instead of the integral around one contour, the half sum of the integral around both. But using a well-known argument, the integral around both cuts equals minus the integral around all the other singularities in the complex θ–plane. i.e. the overall contour integral around the cuts equal the negative of the integral of $(\ldots)$ about all the remaining singularities in the complex θ–plane, *which are poles* at $\theta = -z, w, 1/z, -1/w, \pm i$.

Returning to (2.33), the integral over $C_{1/K}$ involves only the first part of (2.33) the one containing f_K, because the second part does not contain any trapped contour. As for the possible double poles at $\theta = \pm i$, they can at the worst be simple because the double pole of $p_{\pm i}(\theta, w)$ are partly compensated by the zero of $\frac{\theta}{z+\theta} - \frac{1}{1-z\theta}$. Evaluating the residues at the poles we get

$$\oint_{C_{1/K}} d\theta \ldots = -\frac{1}{2} \Bigg\{$$

$$\{\theta = w\} \qquad -\oint \frac{dz}{2\pi i} \frac{1}{z^{n+1}} \oint \frac{dw}{2\pi i} \frac{1}{w^{m+1}} \Bigg[-\frac{f(z)}{f(\frac{1}{w})} \left(\frac{1}{1-zw} - \frac{w}{z+w} \right)$$

$$\cdot \Big(1 - p(z, -w) \Big)$$

$$\{\theta = -z\} \qquad +\frac{f(z)}{f_K(-\frac{1}{z})} \left(\frac{1}{1-zw} - \frac{w}{z+w} \right) \Big(1 - p'(-z, w) \Big) \frac{f(-z)}{f(w)}$$

$$\left\{ \theta = -\frac{1}{w} \right\} \qquad -\frac{f(z)}{f_K(-w)} \left(\frac{1}{1-zw} - \frac{w}{z+w} \right) \frac{f(-\frac{1}{w})}{f(w)} \Big(1 - p(z, -w) \Big)$$

$$\left\{ \theta = \frac{1}{z} \right\} \qquad +\frac{f(z)}{f_K(z)} \frac{f(\frac{1}{z})}{f(w)} \left(\frac{1}{1-zw} - \frac{w}{z+w} \right) \Big(1 - p'(-z, w) \Big)$$

$$\{\theta = \pm i\} \qquad +\ldots \Bigg] \Bigg\} \tag{2.36}$$

where ellipses represent possible contributions of poles at $\theta = \pm i$. The term $\theta = -z, \frac{1}{z}$ cancel exactly the term $\theta = -z$ of (2.34) and the term $\theta = w, -\frac{1}{w}$ cancel the term $\theta = w$ in (2.34). The f factors in each of them become either

$$\gamma f(z) f(w), \qquad \text{or} \qquad \frac{\gamma}{f(z) f(w)}$$

In order to evaluate the contributions of the poles at $\theta = \pm i$ we have to distinguish various cases. If $p = p_i, p' = p_i$, then the contribution of the pole at $\theta = i$ does not appear because,

1) the double pole is partly compensated by the zero of $\frac{\theta}{z+\theta} - \frac{1}{1-z\theta}$, and thus is a simple pole;

2) the residue of this simple pole vanishes due to the factor $f(\theta)^2$, which vanishes as $(\theta - i)^{\frac{4}{3}}$ when $\theta \to i$.

Let us consider next the case $p = p_i, p' = p_{-i}$. The double pole of p_{-i} at $\theta = -i$ is compensated by the zeroes of $\frac{\theta}{z+\theta} - \frac{1}{1-z\theta}$ and $1 - p_i(z, -\theta)$. Therefore the pole disappears.

In the case $p = p_{-i}, p' = p_i$ we have a double pole at $\theta = i$, which is partly compensated by the zero of $\frac{\theta}{z+\theta} - \frac{1}{1-z\theta}$. The remaining simple pole has a zero residue due to $f(\theta)^2$, as in the case $p = p_i, p' = p_i$.

Thus in all three cases just considered, the ellipses at the end of (2.36) correspond to a vanishing contribution.

In the case $p = p_{-i}, p' = p_{-i}$, we have a double pole at $\theta = -i$, which is partly compensated by the usual zero of $\frac{\theta}{z+\theta} - \frac{1}{1-z\theta}$. But the residue of the simple pole is divergent due to $f(\theta)$. Therefore, in this case we have a divergent result.

Finally we can write

$$(\mathcal{U}_{(i)}\,\mathcal{U}_{(i)})_{nm} = (\mathcal{U}_{(i)}\,\mathcal{U}_{(-i)})_{nm} = (\mathcal{U}_{(-i)}\,\mathcal{U}_{(i)})_{nm} \tag{2.37}$$
$$= \oint \frac{dz}{2\pi i} \frac{1}{z^{n+1}} \oint \frac{dw}{2\pi i} \frac{1}{w^{m+1}} \frac{z^2 w^2}{1 - zw} = \begin{cases} \delta_{nm}, & n, m \geq 2 \\ 0, & -1 \leq n \, \text{or} \, m \leq 1 \end{cases}$$

The ss matrix in the r.h.s. of this equation will be denoted by $\mathbf{1}_{ss}$.

On the other hand, $\mathcal{U}_{(-i)}\mathcal{U}_{(-i)}$ is singular.

After twist conjugation we get also

$$\bar{\mathcal{U}}_{(-i)}\,\bar{\mathcal{U}}_{(-i)} = \bar{\mathcal{U}}_{(i)}\,\bar{\mathcal{U}}_{(-i)} = \bar{\mathcal{U}}_{(-i)}\,\bar{\mathcal{U}}_{(i)} = \mathbf{1}_{ss} \tag{2.38}$$

while $\bar{\mathcal{U}}_{(i)}\,\bar{\mathcal{U}}_{(i)}$ is singular.

On the basis of the previous discussion one can better understand the origin of the singularity, mentioned in subsection 2.2, which arises if p_i is simply replaced by p_0. The latter contains both p_i and p_{-i} and we have seen above that when two p_{-i} simultaneously enter into the game we cannot avoid a singularity.

2.5 Fundamental properties of V^{rs}

The calculations relevant to the fundamental properties of the Neumann coefficients for the real vertex V^{rs} are completed in appendix B. To summarize the results obtained, after incorporating those of appendix B, in a compact form, we will use the 3x3 matrix z introduced in section 2.3, and introduce the $\infty \times 3$ matrix u, $u_{n,i} = U_{n,i}$ ($n \geq 2, -1 \geq i \geq 1$), as well as its twist conjugate $\bar{u}$, and an analogous matrix e, $e_{n,i} = E_{n,i}$. Then

$$U^2 = (\mathcal{U} + Z)(\mathcal{U} + Z) = \mathcal{U}^2 + \mathcal{U}Z = \mathbf{1}_{ss} - u\,z, \tag{2.39}$$
$$\bar{U}^2 = (\bar{\mathcal{U}} + Z)(\bar{\mathcal{U}} + Z) = \mathbf{1}_{ss} - \bar{u}\,z \tag{2.40}$$

and

$$E^2 = (\mathcal{E} + Z)(\mathcal{E} + Z) = \mathcal{E}^2 + \mathcal{E}Z = \mathbf{1}_{ss} - e\,z \tag{2.41}$$

Likewise we have

$$E\,U = \mathcal{C}\,U, \qquad U\,E = U\mathcal{C} - \mathcal{C}\,e\,z - u\,z \tag{2.42}$$

i.e., after twisting and combining,

$$E\,\bar{U} = \mathcal{C}\,\bar{U}, \qquad \bar{U}\,E = \bar{U}\mathcal{C} + e\,z - \bar{u}\,z\,\hat{c} \tag{2.43}$$

where $\hat{c}$ is $\mathcal{C}$ reduced to the first 3×3 block.

It was noted in section 2.3 that we could add the 3×3 matrix z to E, U and $\bar{U}$ without changing the three string vertex. We use this freedom to redefine the vertex Neumann matrices. This simple move will dramatically simplify everything.

Let us set

$$E' = E + z, \qquad U' = U + z, \qquad \bar{U}' = \bar{U} + z \tag{2.44}$$

and let us compute U'^2:

$$U'^2 = U^2 + U\,z + z^2 = \mathbf{1}_{ss} - u\,z + u\,z + 1_{3x3} = 1 \tag{2.45}$$

where, now, 1 is the identity matrix in the full range $-1 \leq n, m < \infty$. In the last derivation we have used the fact that $z\,U = 0$. Similarly we can prove that

$$\bar{U}'^2 = 1, \qquad E'^2 = 1 \tag{2.46}$$

Moreover, using again the results of subsection 2.6:

$$E'U' = \mathcal{C}U + 1_{3x3} + e\,z \tag{2.47}$$
$$U'E' = U\mathcal{C} - \mathcal{C}\,e\,z + 1_{3x3} \tag{2.48}$$

Twist–conjugating the second equation we get

$$\bar{U}'E' = \mathcal{C}U + e\,z + 1_{3x3} \tag{2.49}$$

Therefore

$$E'U' = \bar{U}'E' \tag{2.50}$$

Twist–conjugating this

$$E'\bar{U}' = U'E' \tag{2.51}$$

We can now define two types of X matrices, $X_E^{rs} = E'\hat{V}'^{rs}$ and $X'^{rs} = \mathcal{C}\hat{V}'^{rs}$ ($\hat{V}'^{rs} = \hat{V}^{rs} + z\delta_{rs}$),

$$X_E^{rs} = \frac{1}{3}(1 + \bar{\alpha}^{r-s}E'U' + \alpha^{r-s}E'\bar{U}') \tag{2.52}$$

or

$$X'^{rs} = \frac{1}{3}(\mathcal{C}E' + \bar{\alpha}^{r-s}\mathcal{C}U' + \alpha^{r-s}\mathcal{C}\bar{U}') \tag{2.53}$$

The ghost three string vertex Neumann matrices $\hat{V}^{rs}$, obtained from X'^{rs} dropping the z matrix, are those defined in section 1.1, eq. (1.9).

Our first aim is to prove that

$$X_E^{rs} X_E^{r's'} = X_E^{r's'} X_E^{rs} \tag{2.54}$$

and

$$X'^{rs} X'^{r's'} = X'^{r's'} X'^{rs} \tag{2.55}$$

for any r, s, r', s'. For conciseness we write $\alpha^{s-r} = \beta$ and $\alpha^{s'-r'} = \beta'$. To start with, using (2.29) we get

$$X'^{rs} X'^{r's'} = \frac{1}{9}\Big(E'E' + \beta'E'U' + \bar{\beta}'E'\bar{U}' + \beta\bar{U}'E' + \beta\beta'\bar{U}'U' \\ + \beta\bar{\beta}'\bar{U}'^2 + \bar{\beta}U'E' + \bar{\beta}\beta'U'^2 + \bar{\beta}\bar{\beta}'U'\bar{U}'\Big) \tag{2.56}$$

Similarly

$$X'^{r's'} X'^{rs} = \frac{1}{9}\Big(E'E' + \beta E'U' + \bar{\beta}E'\bar{U}' + \beta'\bar{U}'E' + \beta'\beta\bar{U}'U' \\ + \beta'\bar{\beta}\bar{U}'^2 + \bar{\beta}'U'E' + \bar{\beta}'\beta U'^2 + \bar{\beta}'\bar{\beta}U'\bar{U}'\Big) \tag{2.57}$$

The necessary conditions for (2.55) to hold are

$$E'U' = \bar{U}'E', \qquad E'\bar{U}' = U'E', \qquad U'^2 = \bar{U}'^2 \tag{2.58}$$

This is certainly true on the basis of the previous results. In the same way one can prove (2.54). Moreover it is not hard to show

$$\begin{aligned} X_E + X_E^+ + X_E^- &= 1 \\ X_E^+ X_E^- &= X_E^2 - X_E \\ X_E^2 + (X_E^+)^2 + (X_E^-)^2 &= 1 \\ (X_E^+)^3 + (X_E^-)^3 &= 1 + 2X_E^3 - 3X_E^2 \end{aligned} \tag{2.59}$$

The analogous relations for the X' matrices are not as simple. Unfortunately the X_E's are not the matrices that are going to appear in the star product of two string states (see below). In the star product the relevant matrices are the X''s. We have

$$X' - X_E = \mathcal{C}E' - 1 = -1_{3\times3} + \hat{c}z + \hat{c}e \equiv \mathfrak{E} \tag{2.60}$$

while $X_E^\pm = X'^\pm$. It is easy to see that $\mathfrak{E}$ has only two nontrivial columns, precisely the only nonvanishing entries are $\mathfrak{E}_{-1,2n+1} = \mathfrak{E}_{1,2n+1} = -(-1)^n(2n+1)$, $n = -1, 0, 1, \ldots$. Moreover

$\mathfrak{E}$ commutes with all X_E's and X''s matrices, and $\mathfrak{E}^2 = -2\mathfrak{E}$, $\mathfrak{E}X' = X'\mathfrak{E} = -\mathfrak{E}$. Finally one can prove

$$
\begin{aligned}
X' + X'^+ + X'^- &= 1 + \mathfrak{E} \\
X'^+ X'^- &= X'^2 - X' + \mathfrak{E} \\
X'^2 + (X'^+)^2 + (X'^-)^2 &= 1 \\
(X'^+)^3 + (X'^-)^3 &= 1 + 2X'^3 - 3X'^2 - 2\mathfrak{E}
\end{aligned}
\tag{2.61}
$$

2.6 Fundamental properties of $V_{(i)}^{rs}$ and $V_{(-i)}^{rs}$

We proceed in analogy to the previous case. Although the procedure is the same many important details are different and we are forced to repeat the derivations. In the following we consider only $V_{(i)}^{rs}$, because everything concerning $V_{(-i)}^{rs}$ can be obtained by twist–conjugation ($\mathcal{C}V_{(i)}^{rs}\mathcal{C} = V_{(-i)}^{sr}$). Many calculations relevant for the fundamental properties of the Neumann coefficients for the vertex $V_{(i)}^{rs}$ can be found in appendix B. As before we will summarize the results in a compact form by means of the 3x3 matrix z and the $\infty \times 3$ matrix $u_{(i)}$, $u_{(i)n,i} = U_{(i)n,i}$ ($n \geq 2, -1 \geq i \geq 1$), as well as its twist conjugate $\bar{u}_{(i)}$ and the analogous matrix $e_{(i)}$, $e_{(i)n,i} = E_{(i)n,i}$. Then

$$
U_{(-i)}U_{(i)} = (\mathcal{U}_{(-i)} + Z)(\mathcal{U}_{(i)} + Z) = \mathcal{U}_{(-i)}\mathcal{U}_{(i)} + \mathcal{U}_{(-i)}Z = \mathbf{1}_{ss} - u_{(-i)}z, \tag{2.62}
$$

$$
\bar{U}_{(-i)}\bar{U}_{(i)} = (\bar{\mathcal{U}}_{(-i)} + Z)(\bar{\mathcal{U}}_{(i)} + Z) = \mathbf{1}_{ss} - \bar{u}_{(-i)}z \tag{2.63}
$$

and

$$
E_{(-i)}E_{(i)} = (\mathcal{E}_{(-i)} + Z)(\mathcal{E}_{(i)} + Z) = \mathbf{1}_{ss} - e_{(-i)}z \tag{2.64}
$$

Likewise we have

$$
E_{(-i)}U_{(i)} = \mathcal{C}U_{(i)}, \qquad U_{(-i)}E_{(i)} = U_{(-i)}\mathcal{C} + e_{(-i)}z - u_{(-i)}z \tag{2.65}
$$

$$
E_{(-i)}\bar{U}_{(i)} = \mathcal{C}\bar{U}_{(-i)}, \qquad \bar{U}_{(-i)}E_{(i)} = \bar{U}_{(-i)}\mathcal{C} + e_{(-i)}z - \bar{u}_{(-i)}z\,\hat{c} \tag{2.66}
$$

where $\hat{c}$ is $\mathcal{C}$ reduced to the first 3×3 block.

Now, as noted before, we are allowed to add the 3×3 matrix z to the E, U and $\bar{U}$ matrices without changing the three string vertex. We use this freedom to redefine the vertex Neumann matrices. Again, this will simplify everything.

Let us set

$$
E'_{(\pm i)} = E_{(\pm i)} + z, \qquad U'_{(\pm i)} = U_{(\pm i)} + z, \qquad \bar{U}'_{(\pm i)} = \bar{U}_{(\pm i)} + z \tag{2.67}
$$

and let us compute $U'_{(-i)}U'_{(i)}$:

$$
U'_{(-i)}U'_{(i)} = U_{(-i)}U_{(-i)} + U_{(i)}z + z^2 = \mathbf{1}_{ss} - u_{(-i)}z + u_{(-i)}z + 1_{3x3} = 1 \tag{2.68}
$$

where, now, 1 is the identity matrix in the full range $-1 \leq n, m < \infty$. In the last derivation we have used the fact that $z U_{(\pm i)} = 0$. Similarly we can prove that

$$
\bar{U}'_{(-i)}\bar{U}'_{(i)} = 1, \qquad E'_{(-i)}E'_{(i)} = 1 = E'_{(i)}E'_{(-i)} \tag{2.69}
$$

Moreover

$$E'_{(-i)} U'_{(i)} = \bar{U}'_{(-i)} E'_{(i)}, \qquad E'_{(-i)} \bar{U}'_{(i)} = U'_{(-i)} E'_{(i)} \qquad (2.70)$$

As before we can now define two types of X matrices, $X^{rs}_{(i)E} = E'\hat{V}'^{rs}_{(i)}$ and $X'^{rs}_{(i)} = \mathcal{C}\hat{V}'^{rs}_{(i)}$
($\hat{V}'^{rs}_{(i)} = \hat{V}^{rs}_{(i)} + z\delta_{rs}$):

$$X^{rs}_{(i)E} = \frac{1}{3}(1 + \bar{\alpha}^{r-s} E'_{(-i)} U'_{(i)} + \alpha^{r-s} E'_{(-i)} \bar{U}'_{(i)}) \qquad (2.71)$$

or

$$X'^{rs}_{(i)} = \frac{1}{3}(\mathcal{C}E'_{(i)} + \bar{\alpha}^{r-s} \mathcal{C}U'_{(i)} + \alpha^{r-s} \mathcal{C}\bar{U}'_{(i)}) \qquad (2.72)$$

The ghost three string vertex Neumann matrices $\hat{V}^{rs}_{(i)}$, obtained from $X'^{rs}_{(i)}$ dropping the z matrix, are those defined in section 1.1, eq. (1.6).

Using the previous results and the methods of the previous subsection we can prove that

$$X^{rs}_{(i)E} X^{r's'}_{(i)E} = X^{r's'}_{(i)E} X^{rs}_{(i)E} \qquad (2.73)$$

and

$$X'^{rs}_{(i)} X'^{r's'}_{(i)} = X'^{r's'}_{(i)} X'^{rs}_{(i)} \qquad (2.74)$$

for any r, s, r', s'. In addition the $X_{(i)E}$ matrices commute with the $X'_{(i)}$ ones.

Moreover it is not hard to show that

$$\begin{aligned} X_{(i)E} + X^+_{(i)E} + X^-_{(i)E} &= 1 \\ X^+_{(i)E} X^-_{(i)E} &= X^2_{(i)E} - X_{(i)E} \\ X^2_{(i)E} + (X^+_{(i)E})^2 + (X^-_{(i)E})^2 &= 1 \\ (X^+_{(i)E})^3 + (X^-_{(i)E})^3 &= 1 + 2X^3_{(i)E} - 3X^2_{(i)E} \end{aligned} \qquad (2.75)$$

The analogous relations for the $X'_{(i)}$ matrices are not as simple. Unfortunately in the star product the relevant matrices are the $X'_{(i)}$'s. We have

$$X'_{(i)} - X_{(i)E} = -1_{3\times3} + \hat{c}z + \hat{c}e \equiv \mathfrak{E} \qquad (2.76)$$

while $X^{\pm}_{(i)E} = X'^{\pm}_{(i)}$. Moreover $\mathfrak{E}$ commutes with all $X_{(i)E}$'s and $X'_{(i)}$'s matrices, and $\mathfrak{E}^2 = -2\mathfrak{E}$, $\mathfrak{E}X'_{(i)} = X'_{(i)}\mathfrak{E} = -\mathfrak{E}$. Finally one can prove

$$\begin{aligned} X'_{(i)} + X'^+_{(i)} + X'^-_{(i)} &= 1 + \mathfrak{E} \\ X'^+_{(i)} X'^-_{(i)} &= X'^2_{(i)} - X'_{(i)} + \mathfrak{E} \\ X'^2_{(i)} + (X'^+_{(i)})^2 + (X'^-_{(i)})^2 &= 1 \\ (X'^+_{(i)})^3 + (X'^-_{(i)})^3 &= 1 + 2X'^3_{(i)} - 3X'^2_{(i)} - 2\mathfrak{E} \end{aligned} \qquad (2.77)$$

3 Commutators with K_1

The fact that the twisted Neumann matrices of the three strings vertices introduced in the previous section commute opens the way to their diagonalization. The basis we will use is formed by the eigenvectors of the matrix G, which represents together with H^T the operator K_1. The operator K_1 plays a fundamental auxiliary role in SFT and in particular in relation with the three ghost strings vertex. Let us recall the relevant definitions from I.

$$K_1 = \sum_{p,q \geq -1} c_p^\dagger G_{pq} b_q + \sum_{p,q \geq 2} b_p^\dagger H_{pq} c_q - 3c_2 b_{-1} \tag{3.1}$$

where

$$\begin{aligned} G_{pq} &= (p-1)\delta_{p+1,q} + (p+1)\delta_{p-1,q}, \\ H_{pq} &= (p+2)\delta_{p+1,q} + (p-2)\delta_{p-1,q} \end{aligned} \tag{3.2}$$

Therefore G is a square long–legged matrix and H a square short–legged one. In the common overlap we have $G = H^T$. Since we are able to completely solve the spectral problem for G, the latter will be a constant reference in the forthcoming developments.

The generating functions for G and H are

$$G(z,w) = \frac{z - 2w + 3zw^2}{zw(1-zw)^2} \tag{3.3}$$

$$H(z,w) = \frac{wz^2(3 + w^2 - 2zw)}{(1-zw)^2} \tag{3.4}$$

As matrices they have the block structure

$$G = \begin{pmatrix} g_0 & | & 0 \\ -- & & -- \\ \hat{g} & | & \hat{G} \end{pmatrix}, \qquad H^T = \begin{pmatrix} 0 & | & 0 \\ -- & & -- \\ 0 & | & \hat{G} \end{pmatrix}, \tag{3.5}$$

where g_0 is a 3×3 matrix and $\hat{g}$ represents a $\infty \times 3$ matrix with only one entry (the one in position (1,2)) different from zero.

3.1 G and the V^{rs} vertex

Let us consider G from now on. We want to prove that it commutes with $\mathcal{C}U$. Since G anticommutes with $\mathcal{C}$, this is equivalent to compute the anticommutator of G with $U_{(i)}$, which can be easily done both numerically and analytically. Here is the analytic proof

$$\begin{aligned} (\mathcal{U}_{(i)} G)_{nm} &= \oint \frac{dz}{2\pi i} \frac{1}{z^{n+1}} \oint \frac{d\zeta}{2\pi i} \oint \frac{d\theta}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{w^{m+1}} \frac{\zeta\theta}{\zeta\theta - 1} \\ &\quad \cdot \frac{f(z)}{f(\zeta)} \left(\frac{1}{1+z\zeta} - \frac{\zeta}{\zeta - z} \right) \left(1 - p_i(z,\zeta) \right) \frac{\theta - 2w + 3\theta w^2}{w\theta(1 - w\theta)^2} = * \end{aligned} \tag{3.6}$$

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Integrating wrt to θ (pole at $\theta = \frac{1}{\zeta}$)

$$
\begin{aligned}
* = &\oint \frac{dz}{2\pi i} \frac{1}{z^{n+1}} \oint \frac{d\zeta}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{w^{m+1}} \frac{f(z)}{f(\zeta)} \Big(\frac{1}{1+z\zeta} - \frac{\zeta}{\zeta-z} \Big) \\
&\cdot \Big(1 - p_i(z,\zeta) \Big) \frac{\zeta}{w} \frac{1-2w\zeta+3w^2}{(\zeta-w)^2} \\
= &\oint \frac{dz}{2\pi i} \frac{1}{z^{n+1}} \oint \frac{dw}{2\pi i} \frac{1}{w^{m+1}} \bigg[-\frac{z^2}{w} \frac{1-2wz+3w^2}{(z-w)^2} \\
&+ \frac{d}{d\zeta} \Big(\frac{f(z)}{f(\zeta)} \Big(\frac{1}{1+z\zeta} - \frac{\zeta}{\zeta-z} \Big) \Big(1 - p_i(z,\zeta) \Big) \frac{\zeta}{w} (1-2w\zeta+3w^2) \Big)_{\zeta=w} \bigg]
\end{aligned}
\tag{3.7}
$$

The last two terms come from poles at $\zeta = z$ and $\zeta = w$, respectively.

Let us add the ordering term and repeat the same procedure.

$$
\begin{aligned}
(ZG)_{nm} = &\oint \frac{dz}{2\pi i} \frac{1}{z^{n+1}} \oint \frac{d\zeta}{2\pi i} \oint \frac{d\theta}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{w^{m+1}} \frac{\zeta\theta}{\zeta\theta-1} \\
&\cdot \Big(-\frac{z^2}{\zeta} \frac{1}{z-\zeta} \Big) \frac{\theta-2w+3\theta w^2}{w\theta(1-w\theta)^2} \\
= &\oint \frac{dz}{2\pi i} \frac{1}{z^{n+1}} \oint \frac{d\zeta}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{w^{m+1}} \Big(-\frac{z^2}{z-\zeta} \frac{1-2w\zeta+3w^2}{w(\zeta-w)^2} \Big) \\
= &\oint \frac{dz}{2\pi i} \frac{1}{z^{n+1}} \oint \frac{dw}{2\pi i} \frac{1}{w^{m+1}} \bigg[\frac{z^2}{w} \frac{1-2wz+3w^2}{(z-w)^2} - \frac{d}{d\zeta} \Big(\frac{z^2}{w} \frac{1-2w\zeta+3w^2}{z-\zeta} \Big)_{\zeta=w} \bigg]
\end{aligned}
\tag{3.8}
$$

The first piece in the r.h.s. of (3.7) cancels exactly the first piece in the r.h.s. of (3.8), so we have only to evaluate the sum of the two remaining derivatives wrt ζ.

Next let us compute GU. We start with

$$
\begin{aligned}
(GU_{(i)})_{nm} = &\oint \frac{dz}{2\pi i} \frac{1}{z^{n+1}} \oint \frac{d\zeta}{2\pi i} \oint \frac{d\theta}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{w^{m+1}} \frac{\zeta\theta}{\zeta\theta-1} \\
&\cdot \frac{z-2\zeta+3z\zeta^2}{z\zeta(1-z\zeta)^2} \frac{f(\theta)}{f(w)} \Big(\frac{1}{1+w\theta} - \frac{w}{w-\theta} \Big) \Big(1 - p_i(\theta,w) \Big) = *
\end{aligned}
\tag{3.9}
$$

Integrating over ζ we obtain

$$
\begin{aligned}
* = &\oint \frac{dz}{2\pi i} \frac{1}{z^{n+1}} \oint \frac{d\theta}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{w^{m+1}} \frac{z\theta^2-2\theta+3z}{z(\theta-z)^2} \\
&\cdot \frac{f(\theta)}{f(w)} \Big(\frac{1}{1+w\theta} - \frac{w}{w-\theta} \Big) \Big(1 - p_i(\theta,w) \Big) \\
= &\oint \frac{dz}{2\pi i} \frac{1}{z^{n+1}} \oint \frac{dw}{2\pi i} \frac{1}{w^{m+1}} \bigg[\frac{w}{z} \frac{zw^2-2w+3z}{(w-z)^2} \\
&+ \frac{d}{d\theta} \Big(\frac{f(\theta)}{f(w)} \Big(\frac{1}{1+w\theta} - \frac{w}{w-\theta} \Big) \Big(1 - p_i(\theta,w) \Big) \frac{z\theta^2-2\theta+3z}{z} \Big)_{\theta=z} \bigg]
\end{aligned}
\tag{3.10}
$$

The last two terms come from poles at $\theta = w$ and $\theta = z$, respectively.

The ordering term gives,

$$(G\,Z)_{nm} = \oint \frac{dz}{2\pi i}\frac{1}{z^{n+1}} \oint \frac{d\zeta}{2\pi i} \oint \frac{d\theta}{2\pi i} \oint \frac{dw}{2\pi i}\frac{1}{w^{m+1}}\frac{\zeta\theta}{\zeta\theta - 1} \tag{3.11}$$

$$\cdot \frac{z - 2\zeta + 3z\zeta^2}{z\zeta(1-z\zeta)^2}\frac{\theta^2}{w}\frac{-1}{\theta - w}$$

$$= -\oint \frac{dz}{2\pi i}\frac{1}{z^{n+1}} \oint \frac{d\theta}{2\pi i} \oint \frac{dw}{2\pi i}\frac{1}{w^{m+1}}\frac{z\theta^2 - 2\theta + 3z}{z(\theta - z)^2}\frac{\theta^2}{w}\frac{1}{\theta - w}$$

$$= -\oint \frac{dz}{2\pi i}\frac{1}{z^{n+1}} \oint \frac{dw}{2\pi i}\frac{1}{w^{m+1}}\left[\frac{w}{z}\frac{zw^2 - 2w + 3z}{(w-z)^2} + \frac{d}{d\theta}\left(\frac{z\theta^2 - 2\theta + 3z}{zw}\frac{\theta^2}{\theta - w}\right)_{\theta = z}\right]$$

In $G\,U_{(i)}$ the first piece in the r.h.s. of (3.10) cancels the first piece in the r.h.s. of (3.11). What remains is a derivative wrt to ζ in (3.7), (3.8) and wrt to θ in (3.10), (3.8). The derivative in (3.7) gives

$$\frac{f(z)}{f(w)}\left[-\frac{4}{3}i\left(\frac{1}{1+zw} - \frac{w}{w-z}\right)(1 - p_i(z,w)) + \frac{z^2}{w}\frac{(1+w^2)^3}{(w-z)^2(1+wz)^2}\right] \tag{3.12}$$

The derivative in (3.10) gives

$$\frac{f(z)}{f(w)}\left[\frac{4}{3}i\left(\frac{1}{1+zw} - \frac{w}{w-z}\right)(1 - p_i(z,w)) - \frac{z^2}{w}\frac{(1+w^2)^3}{(w-z)^2(1+wz)^2}\right] \tag{3.13}$$

The sum of these two terms vanishes.

The derivative in (3.11) gives

$$\frac{z^2(-1 - 3w^2 + 2wz)}{w(w-z)^2} \tag{3.14}$$

The derivative in (3.8) gives

$$\frac{z^2(1 - 3z^2 + 4wz)}{w(w-z)^2} \tag{3.15}$$

The sum of these two terms is

$$-3\frac{z^2}{w} \tag{3.16}$$

Therefore, apart from this term we get $U_{(i)}G + G\,U_{(i)} = 0$, or

$$[\mathcal{C}U_{(i)}, G]_{nm} = -3\delta_{n,2}\delta_{m,-1} \tag{3.17}$$

This is of course true also for $\mathcal{C}\bar{U}_{(-i)}$.

It is even simpler to prove that

$$[\mathcal{C}E_{(\pm i)}, G]_{nm} = -3\delta_{n,2}\delta_{m,-1} \tag{3.18}$$

The anomaly in the r.h.s. of these commutators is a well–known effect of not having included the last term in the r.h.s. of (3.1) in the definition of G and H. We can easily

cancel this anomaly by adding the 3×3 matrix z to $E, U_{(i)}$ and $\bar{U}_{(-i)}$. For let us compute $[\mathcal{C}U'_{(i)}, G]$. The only change with the commutator $[\mathcal{C}U_{(i)}, G]$ is the addition of $-\hat{g}\, z$. This vanishes everywhere except for the (2,-1) entry, which equals 3. Therefore

$$[\mathcal{C}U'_{(i)}, G] = 0 \tag{3.19}$$

Likewise one can prove that

$$[\mathcal{C}\bar{U}'_{(-i)}, G] = 0, \qquad [\mathcal{C}E'_{(\pm i)}, G] = 0 \tag{3.20}$$

Therefore we conclude that $X^{rs}_{(i)} + \delta^{rs}\hat{c}z$ commute with G.

Summarizing the results of this subsection: the matrices $X'^{rs}, X^{rs}_{(i)}$ as well as the matrix $\mathfrak{E}$ commute with G, therefore they will be diagonal in the bases of eigenvectors of the latter matrix. To conclude let us recall once again that we can always add a matrix z_{ij} to V^{rr}_{ij}, since, as pointed out in section 2.4 this ambiguity is allowed by the formalism.

3.2 G and the $V^{rs}_{(i)}$ vertex

We need to prove that G commutes with $\mathcal{C}U_{(\pm i)}$. The case $U_{(i)}$ has just been analyzed. The case $U_{(-i)}$ requires minor changes. $(\mathcal{U}_{(-i)}G)_{nm}$ is the same as $(\mathcal{U}_{(i)}G)_{nm}$, eqs. (3.6), (3.7) with the simple replacement $p_i \to p_{-i}$, and $G\,(\mathcal{U}_{(-i)})_{nm}$ is the same as $G\,(\mathcal{U}_{(i)})_{nm}$ with the same replacement.

After this replacement the derivative in (3.7) gives

$$\frac{f(z)}{f(w)}\left[-\frac{4}{3}i\left(\frac{1}{1+zw} - \frac{w}{w-z}\right)(1 - p_i(z,w)) + \frac{z^2}{w}\frac{(1+w^2)^3}{(w-z)^2(1+wz)^2}\right]$$

The derivative in (3.10) gives

$$\frac{f(z)}{f(w)}\left[\frac{4}{3}i\left(\frac{1}{1+zw} - \frac{w}{w-z}\right)(1 - p_i(z,w)) - \frac{z^2}{w}\frac{(1+w^2)^3}{(w-z)^2(1+wz)^2}\right]$$

that is, they are the same as before replacement, and the sum of these two terms vanishes.

The derivatives in (3.11) and (3.8) of course remain the same, thus their sum is again $-3\frac{z^2}{w}$.

Therefore we get

$$[\mathcal{C}U_{(-i)}, G]_{nm} = -3\delta_{n,2}\delta_{m,-1} \tag{3.21}$$

This is of course true also for $\mathcal{C}\bar{U}_{(i)}$. Moreover

$$[\mathcal{C}E_{(-i)}, G]_{nm} = -3\delta_{n,2}\delta_{m,-1} \tag{3.22}$$

Again we can eliminate the anomaly in the r.h.s. of these commutators by adding z. Indeed

$$[\mathcal{C}U'_{(-i)}, G] = 0 \tag{3.23}$$

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Likewise

$$[\mathcal{C}\bar{U}'_{(i)}, G] = 0, \qquad [\mathcal{C}\,E'_{(\pm i)}, G] = 0 \tag{3.24}$$

Therefore we conclude that $X^{rs}_{(\pm i)} + \delta^{rs}\hat{c}z$ commute with G.

Summarizing the results of this subsection: the matrices $X'^{rs}_{(\pm i)}, X^{rs}_{(\pm i)E}$ as well as the matrix $\mathfrak{E}$ commute with G, therefore they will be diagonal in the bases of eigenvectors of the latter matrix. Our next purpose is to introduce such bases.

4 The weight 2 and -1 bases

This section is devoted to the bases of eigenfunctions of G. As it turns out the b, c bases introduced in I were incomplete, because only the continuous eigenvalues of G were taken into account, while the discrete ones were disregarded. Once complete bases are introduced, we will be able to write down spectral formulas for the G matrix and for several different Neumann coefficient matrices.

We will also write reconstruction formulas for the A, B, C, D matrices (for their definition see eq. (4.21) below) introduced in I. This analysis was started in [28] (see also [30–32]), where spectral formulas were derived on a heuristic basis. We are now in the condition to clarify to what extent those formulas are valid.

To start with let us recall the definitions of the weight 2 and -1 continuous bases of eigenvectors of G. The unnormalized weight 2 basis is given by

$$f^{(2)}_\kappa(z) = \sum_{n=2} V^{(2)}_n(\kappa)\, z^{n-2} \tag{4.1}$$

in terms of the generating function

$$f^{(2)}_\kappa(z) = \left(\frac{1}{1+z^2}\right)^2 e^{\kappa \arctan(z)} = 1 + \kappa z + \left(\frac{\kappa^2}{2} - 2\right) z^2 + \ldots \tag{4.2}$$

Following [30, 31], (see also appendix B of [3]), we normalize the eigenfunctions as follows

$$\tilde{V}^{(2)}_n(\kappa) = \sqrt{A_2(\kappa)}V^{(2)}_n(\kappa) \tag{4.3}$$

where

$$A_2(\kappa) = \frac{\kappa(\kappa^2 + 4)}{2\sinh\left(\frac{\pi\kappa}{2}\right)}$$

The unnormalized weight -1 basis is given by

$$f^{(-1)}_\kappa(z) = \sum_{n=-1} V^{(-1)}_n(\kappa)\, z^{n+1} \tag{4.4}$$

in terms of the generating function

$$f^{(-1)}_\kappa(z) = (1 + z^2)\, e^{\kappa \arctan(z)} = 1 + \kappa z + \left(\frac{\kappa^2}{2} + 1\right) z^2 + \ldots \tag{4.5}$$

The normalized one is

$$\tilde{V}_n^{(-1)}(\kappa) = \sqrt{A_{-1}(\kappa)} V_n^{(-1)}(\kappa), \qquad \sqrt{A_{-1}(\kappa)} = \mathcal{P} \frac{1}{\kappa} \frac{\sqrt{A_2(\kappa)}}{\kappa^2 + 4} \tag{4.6}$$

where $\mathcal{P}$ denotes principal value. We reported in [3] the 'bi–completeness'

$$\int_{-\infty}^{\infty} d\kappa \, \tilde{V}_n^{(-1)}(\kappa) \, \tilde{V}_m^{(2)}(\kappa) = \delta_{n,m}, \qquad n \geq 2 \tag{4.7}$$

and bi–orthogonality relation

$$\sum_{n=2}^{\infty} \tilde{V}_n^{(-1)}(\kappa) \, \tilde{V}_n^{(2)}(\kappa') = \delta(\kappa, \kappa') \tag{4.8}$$

taking them from [32].

As for the first three elements of the -1 basis, $\tilde{V}_i^{(-1)}(\kappa)$, $i = -1, 0, 1$, they can be expressed in terms of the others (see [32] and appendix B of [3])

$$\tilde{V}_i^{(-1)}(\kappa) = \sum_{n=2}^{\infty} b_{i,n} \, \tilde{V}_n^{(-1)}(\kappa) \tag{4.9}$$

One can easily show that

$$b_{-1,2n+3} = (-1)^n (n+1), \qquad b_{0,2n+2} = (-1)^n, \qquad b_{1,2n+3} = (-1)^n (n+2) \tag{4.10}$$

However the 'bi–completeness' relation (4.7) is not complete. The reason can be understood by studying the spectrum of G. The matrix G looks as follows

$$G = \begin{pmatrix} 0 & -2 & 0 & 0 & 0 & \dots \\ 1 & 0 & -1 & 0 & 0 & \dots \\ 0 & 2 & 0 & 0 & 0 & \dots \\ 0 & 0 & 3 & 0 & 1 & \dots \\ 0 & 0 & 0 & 4 & 0 & \dots \\ . & . & . & . & . & \dots \end{pmatrix} \tag{4.11}$$

It is easy to see that the g_0 matrix (the upper left 3×3 block of G) has left eigenvectors $(1,0,1), (1,\pm 2i, -1)$ with eigenvalues $0, \pm 2i$, respectively. By adding to this eigenvectors a sequence of zeroes in position $2, 3, \dots$ they become left eigenvectors of the full G matrix, i.e.

$$V^{(-1)}(0) = (1,0,1,0,0,0,\dots), \qquad V^{(-1)}(\pm 2i) = (1,\pm 2i, -1, 0, 0, 0, \dots) \tag{4.12}$$

are left eigenvectors of G with eigenvalues 0 and $\pm 2i$, respectively. It is easy to see that they correspond to the vectors $V_n^{(-1)}(\kappa)$ for $\kappa = 0, \pm 2i$, respectively. In other words the discrete eigenvectors are the same as the continuous eigenvectors evaluated at the corresponding eigenvalue in the κ plane.

g_0 has also right eigenvectors, with the same eigenvalues. One can easily check that $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \mp i \\ -1 \end{pmatrix}$ are right eigenvectors with eigenvalues $0, \pm 2i$ respectively. However, in order

to get the right eigenvectors of G it is not enough to add an infinite sequence of zeroes to the eigenvectors of g_0, because of the presence of a nonzero entry in position $(2,1)$ of G.

The problem of finding the right eigenvectors of G can however be solved in an algebraic way. One adds unknowns in position $2, 3, \ldots$ of the vectors and imposes that the resulting vectors be eigenvectors of G with the above discrete eigenvalues. One easily gets

$$v^{(2)}(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ -3 \\ 0 \\ \vdots \end{pmatrix}, \qquad v^{(2)}(\pm 2i) = \begin{pmatrix} 1 \\ \mp i \\ -1 \\ \pm i \\ 1 \\ \mp i \\ \vdots \end{pmatrix} \tag{4.13}$$

More precisely the entries v_n^0 of $v^{(2)}(0)$ are zero for n even and equal $v_{2n+1}^0 = (-1)^n(2n+1) = -(b_{-1,2n+1} + b_{1,2n+1})$ for n odd. The entries $v_n^\pm$ of $V^{(2)}(\pm 2i)$ are $v_{2n}^\pm = \mp i(-1)^n = \pm i b_{0,2n}$ for n even and $v_{2n+1}^\pm = -(-1)^n = b_{1,2n+1} - b_{-1,2n+1}$ for n odd. The b coefficients are the familiar ones

$$b_{0,2n} = -(-1)^n, \qquad b_{1,2n+1} = -(-1)^n(n+1), \qquad b_{-1,2n+1} = -(-1)^n n \tag{4.14}$$

Let us stress that $v^{(2)}(0), v^{(2)}(\pm 2i)$ are *different* from the values taken by the continuous $V^{(2)}(\kappa)$ basis evaluated at $\kappa = 0, \pm 2i$. This is the reason why we use for these discrete eigenvectors different symbols form the continuous ones, while for $V^{(-1)}$ we use the same notation for both.

Next we normalize the discrete eigenvectors as follows

$$\tilde{v}^{(2)}(0) = \frac{1}{\sqrt{2}}v^{(2)}(0), \qquad \tilde{v}^{(2)}(\pm 2i) = \frac{1}{2}v^{(2)}(\pm 2i)$$

$$\tilde{V}^{(-1)}(0) = \frac{1}{\sqrt{2}}V^{(-1)}(0), \quad \tilde{V}^{(-1)}(\pm 2i) = \frac{1}{2}V^{(-1)}(\pm 2i)$$

Using (4.9) it is easy to prove the following orthogonality conditions[1] (we denote by ξ the discrete eigenvalues $0, \pm 2i$)

$$\sum_{n=-1}^{\infty} \tilde{V}_n^{(-1)}(\xi)\tilde{v}_n^{(2)}(\xi') = \delta_{\xi,\xi'} \tag{4.15}$$

$$\sum_{n=-1}^{\infty} \tilde{V}_n^{(-1)}(\kappa)\tilde{v}_n^{(2)}(\xi') = 0$$

$$\sum_{n=-1}^{\infty} \tilde{V}_n^{(-1)}(\xi)\tilde{V}_n^{(2)}(\kappa) = 0$$

$$\sum_{n=-1}^{\infty} \tilde{V}_n^{(-1)}(\kappa)\tilde{V}_n^{(2)}(\kappa') = \delta(\kappa,\kappa')$$

---

[1]These orthogonality conditions certainly hold for κ away from the singularities of the bases normalization factors (see the beginning of this section), but must be otherwise used with extreme care.

To adopt this notation we have added three zeroes to $V_n^{(2)}(\kappa)$ in position $n = -1, 0, 1$. In I, see eq. (5.17), we showed that

$$\sum_{m=-1}^{\infty} V_m^{(-1)} G_{mn} = \kappa \, V_n^{(-1)} \tag{4.16}$$

From the explicit proof it is evident that G is diagonal on $V^{(-1)}(\kappa)$ for *any complex value* of κ. Therefore the second equation in (4.15) holds as long as $\kappa \neq \xi$. More about this issue later on.

Now let us consider the matrix $I_{nm} = \sum_\xi \tilde{v}_n^{(2)}(\xi) \tilde{V}_m^{(-1)}(\xi) + \int d\kappa \, \tilde{V}_n^{(2)}(\kappa) \tilde{V}_m^{(-1)}(\kappa)$. Using (4.15) it is easy to prove that, for instance, $\sum_{m=-1} I_{nm} \tilde{v}_m^{(2)}(\xi) = \tilde{v}_n^{(2)}(\xi)$, etc., both from the right and from the left. Therefore we conclude that

$$\sum_\xi \tilde{v}_n^{(2)}(\xi) \tilde{V}_m^{(-1)}(\xi) + \int_{-\infty}^{\infty} d\kappa \, \tilde{V}_n^{(2)}(\kappa) \tilde{V}_m^{(-1)}(\kappa) = \delta_{nm}, \quad n, m \geq -1 \tag{4.17}$$

This is the correct bi–completeness relation. In this formula it is understood the the integration on κ is along the real axis.

For future use we record

$$v_n^{(2)}(\xi) + \sum_{i=-1}^{1} b_{i,n} \, v_i^{(2)}(\xi) = 0 \tag{4.18}$$

4.0.1 Diagonalization of $\mathfrak{E}$

We have already noticed that the $\mathfrak{E}$ matrix of the previous subsection is diagonal in the basis of G eigenvectors. Indeed one easily realizes that $\mathfrak{E} V^{(2)}(\kappa) = 0 = V^{(-1)}(\kappa) \mathfrak{E}$ for the continuous eigenvectors, while

$$\mathfrak{E} v^{(2)}(0) = -2 v^{(2)}(0), \qquad \mathfrak{E} v^{(2)}(\pm 2i) = 0 \tag{4.19}$$
$$V^{(-1)}(0) \mathfrak{E} = -2 V^{(-1)}(0), \; V^{(-1)}(\pm 2i) \mathfrak{E} = 0$$

for the discrete ones. Therefore the presence of $\mathfrak{E}$ in (2.61) only affects the 0 discrete eigenvalue of G.

4.1 Spectral formulas

Using the spectral representation one can reconstruct G from its eigenvalues and eigenvectors:

$$G_{nm} = \int_{-\infty}^{\infty} d\kappa \, \tilde{V}_n^{(2)}(\kappa) \, \kappa \, \tilde{V}_m^{(-1)}(\kappa) + \sum_\xi \tilde{V}_n^{(2)}(\xi) \, \xi \, \tilde{V}_m^{(-1)}(\xi) \tag{4.20}$$

$$= \int d\kappa_{-\infty}^{\infty} \frac{\kappa}{2 \sinh \frac{\pi \kappa}{2}} V_n^{(2)}(\kappa) V_m^{(-1)}(\kappa) + \frac{i}{2} \left(v_n^{(2)}(2i) \, V_m^{(-1)}(2i) - v_n^{(2)}(-2i) \, V_m^{(-1)}(-2i) \right)$$

For instance, for $-1 \leq i, j \leq 1$ we have

$$G_{ij} = \frac{i}{2} \left(\begin{pmatrix} 1 \\ -i \\ -1 \end{pmatrix} \otimes (1, 2i, -1) - \begin{pmatrix} 1 \\ i \\ -1 \end{pmatrix} \otimes (1, -2i, -1) \right) = \begin{pmatrix} 0 & -2 & 0 \\ 1 & 0 & -1 \\ 0 & 2 & 0 \end{pmatrix}$$

Next, using (4.13),

$$G_{21} = \int d\kappa \frac{\kappa\left(\frac{\kappa^2}{2}+1\right)}{2\sinh\frac{\pi\kappa}{2}} + \frac{i}{2}(i(-1)-(-i)(-1)) = 2+1 = 3$$

Similarly $G_{2,-1} = 0, G_{2,0} = 0$ and so on, as expected.

4.1.1 Properties of the G spectrum

According to formula (5.17) of I (or (4.16) above), formally any value of κ is a continuous eigenvalue of G. Nothing prevents us from extending eqs. (4.15) to complex κ, provided we remain in a strip around the real axis. Indeed for $\kappa = \pm 2in$, with natural n, something happens: the $V^{(-1)}(\kappa)$ basis has only a finite number of nonvanishing terms and the measure in the spectral formula (4.20) above has a simple pole. In fact, if we compute for instance the element G_{21}, we get 3 as above as long as the integration contour stretches from $-\infty$ to $+\infty$ in the strip $|\Im(\kappa)| < 2$, it becomes 7 in the band $2 < |\Im(\kappa)| < 4$, -49 in the band $4 < |\Im(\kappa)| < 6$, etc. These jumps are due exactly to the contributions of the poles: as one moves the contour up or down some poles may remain trapped inside the contour, giving rise to a contribution which equals exactly the corresponding residue.

From this we learn that, unless we do not want to correct the results by hand each time, the good region for the spectral formula of G is $|\Im(\kappa)| < 2$.

4.1.2 The reconstruction of A, B, C, D

The A, B, C, D matrices are defined by the relation

$$\mathcal{L}_0^{(g)} + \mathcal{L}_0^{(g)\dagger} \equiv c_M^\dagger A_{Mn} b_n^\dagger + c_M^\dagger C_{MN} b_N + b_m^\dagger D_{mn} c_n - c_m B_{mN} b_N \qquad (4.21)$$

They were explicity calculated in I. Here we want to discuss their reconstruction formulas. In [28] we numerically proved the reconstruction formulas (4.28) and (4.31) for the bulk $\tilde{A}$ and D^T matrices, using boundary data. We show below that the boundary data information is contained in the discrete basis.

To start with let us propose the spectral formulas:

$$\tilde{A}_{nm} = \int_{-\infty}^{\infty} d\kappa \tilde{V}_n^{(2)}(\kappa)\mathfrak{a}(\kappa)\tilde{V}_m^{(-1)}(\kappa) + \sum_\xi \tilde{v}_n^{(2)}(\xi)\mathfrak{a}(\xi)\tilde{V}_m^{(-1)}(\xi) \qquad (4.22)$$

$$C_{nm} = \int_{-\infty}^{\infty} d\kappa \tilde{V}_n^{(2)}(\kappa)\mathfrak{c}(\kappa)\tilde{V}_m^{(-1)}(\kappa) + \sum_\xi \tilde{v}_n^{(2)}(\xi)\mathfrak{c}(\xi)\tilde{V}_m^{(-1)}(\xi) \qquad (4.23)$$

Let us recall from I the continuous eigenvalues of $\tilde{A}$ and C (see also the re–derivation of these formulas in appendix C)

$$\mathfrak{a}(\kappa) = \frac{\pi\kappa}{2}\frac{1}{\sinh\left(\frac{\pi\kappa}{2}\right)}, \qquad \mathfrak{c}(\kappa) = \frac{\pi\kappa}{2}\frac{\cosh\left(\frac{\pi\kappa}{2}\right)}{\sinh\left(\frac{\pi\kappa}{2}\right)} \qquad (4.24)$$

and notice that $\mathfrak{a}(\kappa)$ becomes singular on the discrete points of the spectrum $\kappa = \pm 2i$. Let us assume for the time being that the discrete eigenvalues of A coincide with the continuous ones evaluated at ξ (this is not obvious and will be justified later on).

Taking an expansion about ξ we have

$$\mathfrak{a}(x) = 1 + O(x^2), \qquad \mathfrak{a}(x \pm 2i) = \mp\frac{2i}{x} - 1 + O(x) \tag{4.25}$$

$$\mathfrak{c}(x) = 1 + O(x^2), \qquad \mathfrak{c}(x \pm 2i) = \pm\frac{2i}{x} + 1 + O(x) \tag{4.26}$$

for small x. $V_n^{(2)}(\xi)$ and $V_n^{(-1)}(\xi)$ have been defined above. In particular all entries of $V_n^{(-1)}(\xi)$ vanish in positions $n \geq 2$. We have already remarked that their values coincide with the limit of $V_n^{(-1)}(\kappa)$ when $\kappa \to \xi$. If we use this definition the vanishing of all entries in positions $n \geq 2$ is true only in the limit $x \to 0$, which is enough in general, but not in (4.22) and (4.23), where these zeroes are needed to cancel the poles in (4.25), (4.26). More precisely we have

$$V_n^{(-1)}(0) = \oint \frac{dz}{2\pi i} (1 + z^2) \frac{1}{z^{n+2}} = \delta_{n,1} + \delta_{n,-1} \tag{4.27}$$

and

$$\begin{aligned}
V_n^{(-1)}(2i + x) &= \oint \frac{dz}{2\pi i} (1 + z^2) \left(\frac{1 + iz}{1 - iz}\right)^{1 + \frac{x}{2i}} \frac{1}{z^{n+2}} \\
&\approx \oint \frac{dz}{2\pi i} (1 + iz)^2 \frac{1}{z^{n+2}} + \frac{x}{2i} \oint \frac{dz}{2\pi i} (1 + iz)^2 \ln\left(\frac{1 + iz}{1 - iz}\right) \frac{1}{z^{n+2}} \\
&= \delta_{n,-1} + 2i\,\delta_{n,0} - \delta_{n,1} + \frac{x}{2i} \left(-2i\,A_{0,n} + A_{-1,n} - A_{1,n} + 2i\delta_{n,0} - 4\delta_{n,1}\right)
\end{aligned} \tag{4.28}$$

Therefore, for $n \geq 2$,

$$V_n^{(-1)}(2i + x) \approx \frac{x}{2i} \left(-2i\,A_{0,n} + A_{-1,n} - A_{1,n}\right) \tag{4.29}$$

In a similar way one can prove that

$$V_n^{(-1)}(-2i + x) \approx \delta_{n,-1} - 2i\delta_{n,0} - \delta_{n,1} + \frac{x}{2i} \left(-2i\,A_{0,n} - A_{-1,n} + A_{1,n} + 2i\delta_{n,0} + 4\delta_{n,1}\right)$$

i.e., for $n \geq 2$

$$V_n^{(-1)}(-2i + x) \approx \frac{x}{2i} \left(-2i\,A_{0,n} - A_{-1,n} + A_{1,n}\right) \tag{4.30}$$

We recall that $A_{0,n}$ vanishes for odd n while $A_{1,n} = -A_{-1,n}$ vanishes for even n. Now

$$\begin{aligned}
\sum_\xi \tilde{v}_n^{(2)}(\xi)\mathfrak{a}(\xi)\tilde{V}_m^{(-1)}(\xi) \\
&= \lim_{x \to 0} \left(\mathfrak{a}(x + 2i)\tilde{v}_n^{(2)}(2i)\tilde{V}_m^{(-1)}(x + 2i) + \mathfrak{a}(x - 2i)\tilde{v}_n^{(2)}(-2i)\tilde{V}_m^{(-1)}(x - 2i)\right) \\
&= -\frac{1}{4}(-2i\,A_{0,2m} + A_{-1,2m+1} - A_{1,2m+1})(ib_{0,2n} + b_{1,2n+1} - b_{-1,2n+1}) \\
&\quad + \frac{1}{4}(-2i\,A_{0,2m} - A_{-1,2m+1} + A_{1,2m+1})(ib_{0,2n} + b_{1,2n+1} - b_{-1,2n+1})
\end{aligned}$$

Therefore

$$\sum_\xi \tilde{V}_n^{(2)}(\xi)\mathfrak{a}(\xi)\tilde{V}_m^{(-1)}(\xi) = -b_{0,2n}A_{0,2m} + A_{1,2n+1}(b_{1,2n+1} - b_{-1,2n+1})$$

$$= -\sum_{a=-1,0,1} b_{a,n}\tilde{A}_{a,m}$$

Finally

$$\tilde{A}_{nm} = \int d\kappa \tilde{V}_n^{(2)}(\kappa)\mathfrak{a}(\kappa)\tilde{V}_m^{(-1)}(\kappa) - \sum_{a=-1,0,1} b_{a,n}\tilde{A}_{a,m} \qquad (4.31)$$

This is precisely formula (4.28) of [28].

Similarly, for $a = -1, 0, 1$ and $m \geq 2$, using the same equations and the fact that $V_a^{(2)}(\kappa) = 0$, we find

$$\tilde{A}_{a,m} = \sum_\xi \tilde{v}_a^{(2)}(\xi)\mathfrak{a}(\xi)V_m^{(-1)}(\xi)$$

$$= \lim_{x\to 0}\left(\mathfrak{a}(x+2i)\tilde{v}_a^{(2)}(2i)\tilde{V}_m^{(-1)}(x+2i) + \mathfrak{a}(x-2i)\tilde{v}_a^{(2)}(-2i)\tilde{V}_m^{(-1)}(x-2i)\right)$$

$$= \left(-\frac{1}{4}\begin{pmatrix} 1 \\ -i \\ -1 \end{pmatrix}(-2i\,A_{0,2m} + A_{-1,2m+1} - A_{1,2m+1})\right.$$

$$\left. + \begin{pmatrix} 1 \\ i \\ -1 \end{pmatrix}(-2i\,A_{0,2m} - A_{-1,2m+1} + A_{1,2m+1})\right) \qquad (4.32)$$

This means

$$\tilde{A}_{-1,m} = -A_{-1,m}, \qquad \tilde{A}_{0,m} = A_{0,m}, \qquad \tilde{A}_{1,m} = -A_{1,m} \qquad (4.33)$$

This completes the recostruction of the matrix A, including the first three rows, which in [28] were called *boundary data*. These boundary data turn out in fact to be stored in the discrete basis.

The previous result confirms that the guess for $\mathfrak{a}(x \pm 2i)$ was correct, but it does not say anything about the $\xi = 0$ discrete eigenvalue of A (the latter has not been used in the previuous derivation).

Using the same method it is easy to prove from (4.23) that

$$C_{nm} = \int d\kappa \tilde{V}_n^{(2)}(\kappa)\mathfrak{c}(\kappa)\tilde{V}_m^{(-1)}(\kappa) - \sum_{a=-1,0,1} b_{a,n}\,C_{a,m} \qquad (4.34)$$

for $n, m \geq 2$. And, using (4.23) for $a = -1, 0, 1$ and $m \geq 2$, one can show that

$$C_{-1,m} = -A_{1,m}, \qquad C_{0,m} = -A_{0,m}, \qquad C_{1,m} = -A_{-1,m}$$

i.e

$$C_{a,m} = -A_{-a,m} \qquad (4.35)$$

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This is another set of boundary data of [28] which is contained in the discrete basis. Again this confirm the validity of (4.26) (except for $\xi = 0$).

It remains for us to reconstruct the values of $B_{n,i}$ and $C_{n,i}$. Applying flatly the same formulas above we obtain divergent results, because the divergences of (4.25) and (4.26) are not compensated by vanishing basis vectors. As this does not interfere with the subsequent developments we leave this problem open. Let us simply summarize the following facts: *(eq. 4.22) is a good representation of A, provided we supplement it with three columns of zeroes; it of course provides a good representation for the bulk of B; eq. (4.23) is a good representation for C excluding the first three columns and is a good representation for D^T if we limit ourselves to the bulk of eq. (4.23).*

5 Reconstruction of X, $X^{\pm}$, $X_{(i)}$, $X_{(i)}^{rs}$

The first important test of the formalism is the reconstruction of the three string vertex Neumann coefficient matrices. It is convenient to start from the real (average) vertex.

The definition of $\mathcal{C}X = V^{rr}$, eq. (1.9), is

$$V_{nm}^{rr} = \frac{1}{2} \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^{n-1}} \frac{1}{w^{m+2}} \left(\frac{4}{3} i \frac{1+w^2}{(1+z^2)^2} \frac{f(z)+f(w)}{f(z)-f(w)} - \frac{1}{z-w} \right) \tag{5.1}$$

This must be compared with the reconstruction formula

$$X'_{nm} = \int d\kappa \tilde{V}_n^{(2)}(\kappa) \mathfrak{r}(\kappa) \tilde{V}_m^{(-1)}(\kappa) + \sum_{\xi} \tilde{v}_n^{(2)}(\xi) \mathfrak{r}(\xi) \tilde{V}_m^{(-1)}(\xi) \tag{5.2}$$

In order to make the comparison we have to know the eigenvalues. The continuous eigenvalue of the wedge states from the KP equation, [48] derived in I (see also appendix C), are

$$\mathfrak{t}_t(\kappa) = \frac{\sinh\left(\frac{\pi\kappa}{4}(2-t)\right)}{\sinh\left(\frac{\pi\kappa}{4}t\right)} \tag{5.3}$$

In particular for the case $n = 3$, which must coincide with X', this gives

$$\mathfrak{r}(\kappa) = -\frac{\sinh\left(\frac{\pi\kappa}{4}\right)}{\sinh\left(\frac{3\pi\kappa}{4}\right)} \tag{5.4}$$

We will take this as the appropriate continuous eigenvalue for X'. Next we face the problem of the discrete eigenvalues $\mathfrak{r}(\xi)$. Evaluating the continuous eigenvalue at the discrete points of the spectrum we would get $\mathfrak{r}(0) = -\frac{1}{3}, \mathfrak{r}(\pm 2i) = 1$, but it turns out that this choice is wrong. The appropriate discrete eigenvalues turn out to be (see appendix C for a justification)

$$\mathfrak{r}(0) = -1, \qquad \mathfrak{r}(\pm 2i) = 1 \tag{5.5}$$

Using (5.2), (5.5) we get immediately $X'_{i,m} = 0, \quad m \geq 2$, because $V_i^{(2)}(\kappa) = 0$ and $V_m^{(-1)}(\xi) = 0$, for $x \to 0$, and no singularity to be compensated. Instead

$$
\begin{aligned}
X'_{ij} &= 0 - \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \otimes (1,0,1) + \frac{1}{4} \left(\begin{pmatrix} 1 \\ -i \\ -1 \end{pmatrix} \otimes (1,2i,-1) + \begin{pmatrix} 1 \\ i \\ -1 \end{pmatrix} \otimes (1,-2i,-1) \right) \\
&= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}
\end{aligned}
\tag{5.6}
$$

X, given by (5.1), has vanishing first three rows. Therefore the matrix in (5.6) corresponds rather to the matrix z discussed in section 2.4. But we have already seen that to the Neumann matrices of the left (ghost number 3) vertex we can always add a matrix like z.

Let us see now a sample of the other entries

$$
\begin{aligned}
X_{2,1} &= 0 + \frac{1}{4}(i \cdot 1 - i \cdot 1) = 0 \\
X_{2,0} &= -\int d\kappa \frac{\kappa}{2\sinh\left(\frac{\pi\kappa}{2}\right)} \frac{\sinh\left(\frac{\pi\kappa}{4}\right)}{\sinh\left(\frac{3\pi\kappa}{4}\right)} + \frac{1}{4}(i \cdot 2i - i \cdot (-2i)) = -\frac{32}{27} \\
X_{3,-1} &= -\int d\kappa \frac{\kappa}{2\sinh\left(\frac{\pi\kappa}{2}\right)} \frac{\sinh\left(\frac{\pi\kappa}{4}\right)}{\sinh\left(\frac{3\pi\kappa}{4}\right)} - \frac{1}{2}(-3)1 + \frac{1}{4}(1 \cdot 1 + 1 \cdot 1) = \frac{49}{27} \\
X_{4,0} &= -\int d\kappa \frac{\kappa}{2\sinh\left(\frac{\pi\kappa}{2}\right)} \frac{\kappa^2 - 1}{2} \frac{\sinh\left(\frac{\pi\kappa}{4}\right)}{\sinh\left(\frac{3\pi\kappa}{4}\right)} + \frac{1}{4}(-i \cdot 2i + i \cdot (-2i)) = \frac{320}{243}
\end{aligned}
$$

and in the bulk, where only the continuous spectrum contributes,

$$
X_{3,3} = -\int d\kappa \frac{\kappa}{2\sinh\left(\frac{\pi\kappa}{2}\right)} \frac{\kappa^2(\kappa^2 + 4)}{24} \frac{\sinh\left(\frac{\pi\kappa}{4}\right)}{\sinh\left(\frac{3\pi\kappa}{4}\right)} = -\frac{541}{19683}
$$

and so on. These are precisely the values expected from (5.1).

Analogously, one can reconstruct the $X^\pm$ matrices. In that case, the discrete spectrum does not contribute, that is, $\mathfrak{x}^\pm(\xi) = 0$ for $\xi = 0, \pm 2i$. This implies, correctly, that $X^\pm_{ij} = 0$. For the continuous spectrum, we use the same expressions as for the matter part, see [50]:

$$
\mathfrak{x}^+(\kappa) = -(1 + e^{\frac{\pi\kappa}{2}})\mathfrak{x}(\kappa) \quad ; \quad \mathfrak{x}^-(\kappa) = -(1 + e^{-\frac{\pi\kappa}{2}})\mathfrak{x}(\kappa)
\tag{5.7}
$$

Here is a sample of components of X^+:[2]

$$X_{2,0}^+ = \int d\kappa \frac{\mathfrak{x}^+(\kappa)\kappa}{2\sinh(\frac{\pi\kappa}{2})} = \frac{16}{27}$$

$$X_{3,-1}^+ = \int d\kappa \frac{\mathfrak{x}^+(\kappa)\kappa}{2\sinh(\frac{\pi\kappa}{2})} = \frac{16}{27}$$

$$X_{3,0}^+ = \int d\kappa \frac{\mathfrak{x}^+(\kappa)\kappa^2}{2\sinh(\frac{\pi\kappa}{2})} = \frac{64}{81\sqrt{3}}$$

$$X_{4,0}^+ = \int d\kappa \frac{\mathfrak{x}^+(\kappa)\kappa}{2\sinh(\frac{\pi\kappa}{2})} \left(\frac{\kappa^2}{2} - 2\right) = -\frac{160}{243}$$

$$X_{2,2}^+ = \int d\kappa \frac{\mathfrak{x}^+(\kappa)}{2\sinh(\frac{\pi\kappa}{2})} \left(\frac{\kappa^3}{6} + \frac{2\kappa}{3}\right) = \frac{416}{729}$$

$$X_{3,2}^+ = \int d\kappa \frac{\mathfrak{x}^+(\kappa)\kappa}{2\sinh(\frac{\pi\kappa}{2})} \left(\frac{\kappa^3}{6} + \frac{2\kappa}{3}\right) = \frac{896}{729\sqrt{3}}$$

For X^- one can do the same using $\mathfrak{x}^-(\kappa)$ and see that it also works perfectly.

Let us conclude with some remarks. In the previous section we have seen that the twisted Neumann matrices X'^{rs} commute with G, and thus are diagonal in the bases of its eigenvectors. In this section we have written down such bases and we have shown that by their means we can construct spectral representations of X'^{rs} which faithfully reconstruct the latter. We remark that it is easier to identify the discrete eigenvalues of X'^{rs} by this indirect method rather than by a direct approach.

Let us consider now $X_{(i)}$. This matrix is obtained by twisting

$$V_{(i)nm}^{rr} = \frac{1}{2} \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^{n-1}} \frac{1}{w^{m+2}} \left(\frac{4}{3}i\frac{1+w^2}{(1+z^2)^2}\frac{1}{f(z)-f(w)}\left(\frac{f(w)}{f(z)}\right)^3 - \frac{1}{z-w}\right) \quad (5.8)$$

The corresponding $X_{(i)} = \mathcal{C}V_{(i)}^{rr}$ matrix can be reconstructed from the matrix X above by adding additional contributions (the primed matrices are obtained by adding z)

$$X_{(i)nm} = X_{nm} - \frac{4}{3}i V_n^{(2)}\left(\frac{4}{3}i\right) V_m^{(-1)}\left(\frac{4}{3}i\right) - \frac{2}{3}i V_n^{(2)}(0) V_m^{(-1)}(0) \quad (5.9)$$

The last addend affects only the first three columns. In this equation $V^{(2)}$, as well as $V^{(-1)}$, stands for the continuous basis evaluated at the corresponding points. Let us see some examples of the validity of (5.9)

$$(2,0) : \frac{16}{27} = -\frac{32}{27} + \frac{16}{9} + 0$$

$$(2,-1) : -2i = 0 - \frac{4}{3}i - \frac{2}{3}i$$

$$(3,0) : \frac{64}{27}i = 0 + \frac{64}{27}i + 0$$

$$(3,3) : -\frac{6301}{19683} = -\frac{541}{19683} - \frac{640}{2187} + 0$$

---

[2]In computing some of the components, for example, $X_{2,1}^+$, $X_{2,-1}^+$ and $X_{4,-1}^+$, the integrals must be regularized. This has been done using the principal value prescription.

In a similar way one can deal with $X_{(i)}^{\pm}$, see appendix D.

In order to understand the origin of the correction in (5.9) with respect to (5.2) let us return to the latter, which is the classical spectral formula one would expect, i.e. the summation over the eigenvalues (both continuous and discrete) multiplied by the appropriate eigenprojectors. In that formula the continuous eigenvalues are real. However we have noticed above that the continuous eigenvalues may as well be complex. Therefore we could consider the spectral formula with a contour away from the real axis. If there are poles between the new and the old contour the final results will be different. This is precisely what happens in the passage from (5.2) to (5.9). The difference corresponds to a modification of the integration contour over the continuous spectrum.

The continuous part of the spectrum is

$$\Delta X_{nm} = \int d\kappa \, \mu(3,\kappa) V_n^{(2)}(\kappa) V_m^{(-1)}(\kappa) \tag{5.10}$$

where the measure is

$$\mu(3,\kappa) = \frac{\mathfrak{r}(\kappa)}{2\sinh\left(\frac{\pi\kappa}{2}\right)} = -\frac{1}{2\sinh\left(\frac{\pi\kappa}{2}\right)} \frac{\sinh\left(\frac{\pi\kappa}{4}\right)}{\sinh\left(\frac{3\pi\kappa}{4}\right)} \tag{5.11}$$

This measure has poles at $\kappa = \pm\frac{4}{3}in$ for natural n. If in (5.10) the integration contour is along the real axis we get back X_{nm}. But let us suppose that

$$\Delta X_{nm} = \int_{C_1} d\kappa \, \mu(3,\kappa) V_n^{(2)}(\kappa) V_m^{(-1)}(\kappa) \tag{5.12}$$

where C_1 is a straight contour from $-\infty$ to $+\infty$ with $\frac{4}{3} < \Im(\kappa) < 2$. If we move the upper contour toward the real axis we are bound to meet two poles of the measure, one at $\kappa = \kappa_1 = \frac{4i}{3}$ and another at $\kappa = \kappa_0 = 0$. So finally we obtain the usual integral along the real axis (which corresponds to X) plus two contributions from the two poles that remain trapped inside the contour. The latter are clockwise oriented, so we have to change the sign when calculating the residues.

It is easy to show that near the poles κ_i, see (7.10),

$$\mu(\kappa_1 + x) \approx \frac{2}{3}\frac{1}{\pi x}, \qquad \mu(\kappa_0 + x) \approx \left(\frac{2}{3} - 1\right)\frac{1}{\pi x}$$

where we have kept distinct the contribution represented by -1 for a reason that will become clear later on. Therefore

$$-\oint_{\text{poles}} d\kappa \, \mu(3,\kappa) V_n^{(2)}(\kappa) V_m^{(-1)}(\kappa) \tag{5.13}$$
$$= -\oint dx \, \frac{4i}{6\pi i}\frac{1}{x} \, V_n^{(2)}(\kappa_1 + x) V_m^{(-1)}(\kappa_1 + x) - \oint dx \, \frac{1}{2}\frac{4i}{6\pi i}\frac{1}{x} \, V_n^{(2)}(x) V_m^{(-1)}(x)$$
$$= -\frac{4i}{3} \, V_n^{(2)}(\kappa_1) V_m^{(-1)}(\kappa_1) - \frac{2i}{3} \, V_n^{(2)}(0) V_m^{(-1)}(0)$$

The contribution of the pole at $\kappa = 0$ has been divided by two because only 'half' pole contributes (this is consistent with the remaining calculations). In this way the contribution (5.13) accounts precisely for the difference between X and $X_{(i)}$, except for the

contribution $i\, V_n^{(2)}(0)V_m^{(-1)}(0)$. Taking it into account we can write

$$X_{(i)nm} = \int_{\Im(\kappa)>\frac{4}{3}} d\kappa\, \mu(3,\kappa)V_n^{(2)}\kappa V_m^{(-1)}(\kappa) - i\, V_n^{(2)}(0)V_m^{(-1)}(0) + (\ldots) \qquad (5.14)$$

where the integration contour runs parallel to the real axis just above the pole at $\kappa = \frac{4i}{3}$. The second piece in the r.h.s. of (5.14) is a 'necessary scar' of that formula we will comment about later on. The omitted terms $(\ldots)$ are the contribution from the discrete spectrum (which is not touched by the shift in the κ integration).

6 The spectral argument

It is time to see our three strings vertex at work. Let us consider the star product of two wedge states as in the r.h.s. of (1.2)

$$|S\rangle = \mathcal{N} \exp\left(c^\dagger S b^\dagger\right)|0\rangle \qquad (6.1)$$

i.e.

$$\langle \hat{V}_3|S_1\rangle|S_2\rangle = \langle \hat{S}_{12}| \qquad (6.2)$$

We remark that the states like (6.1) are defined on the ghost number 0 vacuum $|0\rangle$, while the resulting state in the r.h.s. of (6.2) is defined in the ghost number 3 vacuum $\langle \hat{0}|$. Therefore $\langle \hat{S}_{12}|$ is not yet the star product. We will discuss in III on how to recover $|S_{12}\rangle$.

The matrix $S_{12} = \mathcal{C}T_{12}$ is given by the familiar formula

$$T_{12} = X + (X^+, X^-)\frac{1}{1-\Sigma_{12}\mathcal{V}}\Sigma_{12}\begin{pmatrix} X^- \\ X^+ \end{pmatrix} \qquad (6.3)$$

where

$$\Sigma_{12} = \begin{pmatrix} \mathcal{C}S_1 & 0 \\ 0 & \mathcal{C}S_2 \end{pmatrix}, \qquad \mathcal{V} = \begin{pmatrix} X & X^+ \\ X^- & X \end{pmatrix} \qquad (6.4)$$

In these formulas the matrices $X, X^\pm$ represent $X', X'^\pm$ or $X'_{(\pm i)}, X'^\pm_{(\pm i)}$. As for the matrices $T_1 = \mathcal{C}S_1, T_2, T_{12}$ they are supposed to represent wedge states. The latter, denoted simply by $|n\rangle \equiv |S_n\rangle$, must satisfy the recursive star product formula

$$|n\rangle \star |m\rangle = |n + m - 1\rangle \qquad (6.5)$$

Our purpose in the sequel is to prove that the squeezed states at the r.h.s. of (1.2), when star–multiplied with our three strings ghost vertex, do obey the recursive formula (6.5). To this end we will proceed as follows. After determining (which we have done in the previous section) the eigenvalues of the twisted Neumann matrices of the vertex we will determine those of the squeezed states at the r.h.s. of (1.2), by inferring them from the properties of the $gh = 0$ wedge states via the KP equation. We will show that the recursion relations of the wedge states ensuing from (6.5) are satisfied. Finally we will show how to reconstruct

the ghost $gh = 3$ results of the star product with the appropriate spectral formulas. The $gh = 0$ states corresponding to them will be reconstructed in III. This argument is based on the prejudice that $gh = 3$ and $gh = 0$ wedge Neumann functions have the same continuous and discrete eigenvalues. This fact is not at all obvious a priori. But it will be justified beyond any doubt with the reconstruction formulas of the $gh=0$ states in III.

The argument is anything but simple. To facilitate the comprehension let us for the time being assume that T_1, T_2 commute with $X, X^\pm$ (which is not true!). In such a case, setting $T_1 = T_n$ and $T_2 = T_m$ we would get that, if (6.5) is true, it follows from (6.3) that

$$T_{n+m-1} = \frac{X - (T_n + T_m)X + T_n T_m(1 + \mathfrak{E}) + (T_n + T_m)\mathfrak{E}}{1 - (T_n + T_m)X + T_n T_m(X - \mathfrak{E})} \tag{6.6}$$

Setting $T_2 = 0$ we can write the recursion formula

$$T_{n+1} = \frac{X(1 - T_n) + T_n \mathfrak{E}}{1 - T_n X} \tag{6.7}$$

This result is not true for the matrices but we will show it to be true for their eigenvalues. This is due to the fact that the Neumann matrices of the ghost number 0 wedge states have a subset of eigenvectors in common with G, while the remaining ones are different.[3] Once again this fact will be entirely clear only at the end of III, where it will appear that $gh = 0$ and $gh = 3$ wedge states Neumann matrices have the same spectrum. The next thing to be done therefore is to evaluate the eigenvalues of T_n.

6.1 The recursion relations for eigenvalues

The recursion relations for matrices (6.7) are not expected to hold, but we wish to show them to be true for their eigenvalues. Applying (6.3) to the bases $V^{(2)}(\kappa)$ and $V^{(-1)}(\xi)$ one can see that the continuous eigenvalues must satisfy

$$\mathfrak{t}_{n+1}(\kappa) = \mathfrak{r}(\kappa)\frac{1 - \mathfrak{t}_n(\kappa)}{1 - \mathfrak{t}_n(\kappa)\mathfrak{r}(\kappa)}, \qquad \mathfrak{t}_3(\kappa) = \mathfrak{r}(\kappa) \tag{6.8}$$

while for the discrete eigenvalues one should get

$$\mathfrak{t}_{n+1}(\xi) = \frac{\mathfrak{r}(\xi)(1 - \mathfrak{t}_n(\xi)) - 2\mathfrak{t}_n(\xi)\,\delta_{\xi,0}}{1 - \mathfrak{t}_n(\xi)\mathfrak{r}(\xi)}, \qquad \mathfrak{t}_3(\xi) = \mathfrak{r}(\xi) \tag{6.9}$$

where the -2 addend in the numerator comes from the eigenvalue of $\mathfrak{E}$. The values of $\mathfrak{t}_n(\kappa), \mathfrak{t}_n(\xi)$ are determined in appendix C (using the results of I). $\mathfrak{t}_n(\kappa)$ has already been reported in eq. (5.3), while the discrete eigenvalues are given by

$$\mathfrak{t}_n(\xi = 0) = -1, \qquad \mathfrak{t}_n(\pm 2i) = 1 \tag{6.10}$$

---

[3]Two matrices can of course have the same eigenvalues without commuting. An elementary example is given by the two matrices $M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $M_b = \begin{pmatrix} 1 & b \\ 0 & 2 \end{pmatrix}$ with $b \neq 0$. They have the same eigenvalues but do not commute. Moreover (counting left and right eigenvectors) they have two eigenvectors in common, while the other two are different.

It has been shown in I that (5.3) does indeed satisfy the recursion relation (6.8). It is easy to see that (6.9) is also satisfied by the above found values (6.10), provided one observes the following procedure: one first replaces the values $\mathfrak{x}(\pm 2i) = 1, \mathfrak{x}(0) = -1$ while keeping $\mathfrak{t}_n(\xi)$ generic. After simplifying the expression one inserts the values (6.10). We remark that the presence of the $\mathfrak{E}$ matrix in (6.7) is essential in this respect.

It is well–known that (6.8) can be solved in terms of the sliver eigenvalue, [42, 43]. We repeat here this derivation to stress its uniqueness. We require that $|2\rangle$ coincide with the vacuum $|0\rangle$, both for the matter and the ghost sector. This implies in particular that $\mathfrak{t}_2 = 0$ which entails from (6.8) that $\mathfrak{t}_3 = \mathfrak{x}$, $\mathfrak{t}_4 = \frac{\mathfrak{x}}{1+\mathfrak{x}}$, etc. That is, $\mathfrak{t}_n$ is a uniquely defined function of $\mathfrak{x}$. But $\mathfrak{x}$ can be uniquely expressed in terms of the sliver matrix $\mathfrak{t}$

$$\mathfrak{x} = \frac{\mathfrak{t}}{\mathfrak{t}^2 - \mathfrak{t} + 1} \tag{6.11}$$

a formula whose inverse is well–known, [46, 48]

$$\mathfrak{t} = \frac{1}{2\mathfrak{x}} \left(1 + \mathfrak{x} - \sqrt{(1 - \mathfrak{x})(1 + 3\mathfrak{x})} \right) \tag{6.12}$$

Therefore $\mathfrak{t}_n$ can be expressed as a uniquely defined function of $\mathfrak{t}$. Now consider the formula

$$\mathfrak{t}_n = \frac{\mathfrak{t} + (-\mathfrak{t})^{n-1}}{1 - (-\mathfrak{t})^n}$$

It satisfies (6.8) as well as the condition $\mathfrak{t}_2 = 0$, therefore it is the unique solution to (6.8) we were looking for.

For completeness we recall also the recursion relation for the normalization constants

$$\mathcal{N}_{n+1} = \mathcal{N}_n \mathcal{K} \det (1 - T_n X) \tag{6.13}$$

It is easy to see that the discrete eigenvalues give a vanishing contribution to the determinant, therefore the discussion reduces to the continuous eigenvalues, and this was done in I.

In III we will also give evidence that (when the matter sector is coupled) this overall normalization will be 1.

7 Reconstruction of the dual wedge states

In the previous sections we have defined three strings vertices for the ghost part. We have consequently defined a midpoint–star product. It is not possible to do the star product in a single step, i.e. it is not possible to start from two $gh = 0$ states and end up with the resulting star product as a $gh = 0$ state (in the matter case this is possible up to a bpz conjugation). In this case we must go through a two step process. We first compute the $gh = 3$ state which is the result of the operation in eq. (6.2). The second step consists in computing the $gh = 0$ ket corresponding to this result (this operation turns out to be far more complicated than the simple bpz conjugation of the matter sector).

In the rest of this paper we will be concerned with the first step, while the second will be the subject of III. In the previous section we have calculated the eigenvalues of the

resulting ($gh = 3$) object (which we will call the dual or bra star product wedge state) in the weight 2 and -1 discrete and continuous basis. Now we wish to express this state as a squeezed state in the oscillator form. To this end we resort to the reconstruction formulas, which are nothing but the ordinary spectral formulas in which, however, the integration contour for the continuous spectrum must be specified. As we will see, different contours give different results with different characteristics: they may or may not be surface states and may or may not be BRST invariant.

In parallel with section 5 let us write down the spectral representation for the left $gh = 3$ wedge states:

$$\hat{T}'^{(N)}_{nm} = \int d\kappa \tilde{V}_n^{(2)}(\kappa) \mathfrak{t}_N(\kappa) \tilde{V}_m^{(-1)}(\kappa) + \sum_\xi \tilde{v}_n^{(2)}(\xi) \mathfrak{t}_N(\xi) \tilde{V}_m^{(-1)}(\xi) \tag{7.1}$$

where for practical reasons we have slightly changed the notation: $\hat{T}^{(N)} = \mathcal{C}\hat{S}_N$ and the prime denotes the addition of the z matrix. Let us recall that the discrete eigenvalues are

$$\mathfrak{t}^{(N)}(0) = \mathfrak{r}(0) = -1 \quad ; \quad \mathfrak{t}^{(N)}(\pm 2i) = \mathfrak{r}(\pm 2i) = 1 \tag{7.2}$$

while the continuous eigenvalue is given by (5.3). We thus have

$$\hat{T}^{(N)}_{nm} = \int_{-\infty}^{\infty} d\kappa \frac{\mathfrak{t}^{(N)}(\kappa)}{2\sinh(\frac{\pi\kappa}{2})} V_n^{(2)}(\kappa) V_m^{(-1)}(\kappa) -$$
$$- \frac{1}{2} v_n^{(2)}(0) V_m^{(-1)}(0) + \frac{1}{4}(v_n^{(2)}(2i) V_m^{(-1)}(2i) + v_n^{(2)}(-2i) V_m^{(-1)}(-2i)) \tag{7.3}$$

where the integral is, for the time being, along the real axis.

Let us compute a sample of the entries of $\hat{T}_4$

$$\hat{T}'^{(4)}_{2,0} = \int_{-\infty}^{\infty} d\kappa \frac{\hat{\mathfrak{t}}^{(4)}(\kappa)}{2\sinh(\frac{\pi\kappa}{2})} \kappa + \frac{1}{4}(i \cdot 2i - i \cdot (-2i)) = -\frac{1}{4} - 1 = -\frac{5}{4}$$

$$\hat{T}'^{(4)}_{3,-1} = \int_{-\infty}^{\infty} d\kappa \frac{\hat{\mathfrak{t}}^{(4)}(\kappa)}{2\sinh(\frac{\pi\kappa}{2})} \kappa - \frac{1}{2} \cdot (-3) + \frac{1}{4}(1+1) = -\frac{1}{4} + 2 = \frac{7}{4}$$

$$\hat{T}'^{(4)}_{4,0} = \int_{-\infty}^{\infty} d\kappa \frac{\hat{\mathfrak{t}}^{(4)}(\kappa)}{2\sinh(\frac{\pi\kappa}{2})} \kappa \left(\frac{\kappa^2}{2} - 2\right) + \frac{1}{4}((-i) \cdot 2i + i \cdot (-2i)) = \frac{7}{16} + 1 = \frac{23}{16}$$

$$\hat{T}'^{(4)}_{2,2} = \int_{-\infty}^{\infty} d\kappa \frac{\hat{\mathfrak{t}}^{(4)}(\kappa)}{2\sinh(\frac{\pi\kappa}{2})} \left(\frac{\kappa^3}{6} + \frac{2\kappa}{3}\right) = -\frac{3}{16}$$

$$\hat{T}'^{(4)}_{3,3} = \int_{-\infty}^{\infty} d\kappa \frac{\hat{\mathfrak{t}}^{(4)}(\kappa)}{2\sinh(\frac{\pi\kappa}{2})} \kappa \left(\frac{\kappa^4}{24} + \frac{\kappa^2}{6}\right) = -\frac{1}{32}$$

These perfectly agree with the formula for the Neumann coefficients of the left $gh = 3$ states

$$\langle \hat{n} | = \langle \hat{0} | e^{-c_p \hat{S}^{(n)}_{pM} b_M} \tag{7.4}$$

where

$$\hat{S}^{(n)}_{pM} = \oint_0 \frac{dz}{2\pi i} \oint_0 \frac{dw}{2\pi i} \frac{1}{z^{p-1}} \frac{1}{w^{M+2}} \left(\frac{2i}{n} \frac{1+w^2}{(1+z^2)^2} \frac{f_n(z) + f_n(w)}{f_n(z) - f_n(w)} - \frac{1}{z-w}\right) \tag{7.5}$$

and

$$f_n(z) = \left(\frac{1 + iz}{1 - iz}\right)^{\frac{2}{n}} \tag{7.6}$$

The only exception is the insertion of the z matrix in the upper left corner (which, however, can always be done due to an intrinsic ambiguity of the oscillator formalism, as explained in section 2.4).

A similar numerical agreement has been checked also for $\hat{T}^{(5)}$ and higher states. The reconstruction formula has given us back squeezed states that belong to the same family as the average three vertex (see section 5). These states however are not surface states with insertions, which creates problems with BRST invariance (see appendix A and especially III for a discussion of these issues).

7.1 The dual wedge states with $Y(i)$ insertion

In order to get BRST invariant surface states as $gh = 3$ wedge states we have to change the integration contour. To this end we follow the recipe of the second part of section 5. That is we use again (7.1), but redefine the integration contour over the continuous spectrum

$$(T_c^{(N)})_{nm} = \int_{C_N} d\kappa\, \tilde{V}_n^{(2)}(\kappa) \mathsf{t}_N(\kappa) \tilde{V}_m^{(-1)}(\kappa) = \int_{C_N} d\kappa\, \mu(N, \kappa)\, V_n^{(2)}(\kappa) V_m^{(-1)}(\kappa) \tag{7.7}$$

where the subscript c stands for the continuous part of the spectral formula, C_N is the contour to be specified and the measure is

$$\mu(N, \kappa) = \frac{\mathsf{t}_N(\kappa)}{2\sinh(\frac{\pi\kappa}{2})} = \frac{1}{2\sinh(\frac{\pi\kappa}{2})} \frac{\sinh\left(\frac{\pi\kappa}{4}(2 - N)\right)}{\sinh\left(\frac{\pi\kappa}{4}N\right)} \tag{7.8}$$

This measure has poles at $\kappa = \pm\frac{4in}{N}$ for natural n. If in (7.1) the integration contour is along the real axis and we move it up, we are bound to meet the first pole at $\kappa = \kappa_1 = \frac{4i}{N}$. In (7.7) the contour C_N stretches from $-\infty$ to ∞ in the upper κ plane with $\Im(\kappa)$ just above $\frac{4}{N}$. This traps two poles of $\mu(N, \kappa)$, i.e. the poles at $\kappa \equiv \kappa_1 = \frac{4i}{N}$ and $\kappa = \kappa_0 = 0$, lying between this contour and the real axis. Therefore the integral over C_N reduces to the usual integral along the real axis plus the contributions of clockwise oriented contours around $\kappa = \kappa_1$ and κ_0, i.e.

$$(T_c^{(N)})_{nm} = \int_{-\infty}^{\infty} d\kappa \mu(N, \kappa) V_n^{(2)}(\kappa) V_m^{(-1)}(\kappa) \tag{7.9}$$

$$- \oint_{C_{\kappa_1}} d\kappa \mu(N, \kappa) V_n^{(2)}(\kappa) V_m^{(-1)}(\kappa) - \frac{1}{2} \oint_{C_{\kappa_0}} d\kappa \mu(N, \kappa) V_n^{(2)}(\kappa) V_m^{(-1)}(\kappa)$$

where C_{κ_1} and C_{κ_0} are small anticlockwise contours around κ_1 and κ_0, respectively (taking half of the latter contribution for the reason explained in section 5). Let us write $\kappa = \frac{4i}{N}n + x$ for small x. It is easy to show that near the pole and for $N \neq 2$

$$\mu(N, \kappa) \approx \begin{cases} \frac{2}{\pi N} \frac{1}{x}, & \sin\left(\frac{2\pi n}{N}\right) \neq 0 \\ \frac{1}{\pi N} \frac{1}{x}(2 - N), & \sin\left(\frac{2\pi n}{N}\right) = 0 \end{cases} \tag{7.10}$$

When $N = 2$ the measure $\mu(2, \kappa) = 0$, as it should be. However, in that case the relevant measure becomes $\mu(2, \kappa) = \frac{1}{2\sinh(\frac{\pi\kappa}{2})}$ and the poles are at $\kappa = 2in$, and near them we have

$$\mu(2, \kappa) \approx \frac{(-1)^n}{\pi x} \tag{7.11}$$

Returning to $N \neq 2$ we have, for instance,

$$\oint_{C_{\kappa_1}} d\kappa \mu(N, \kappa) V_n^{(2)}(\kappa) V_m^{(-1)}(\kappa) = \oint dx \frac{4i}{2\pi i N} \frac{1}{x} V_n^{(2)}(\kappa_1) V_m^{(-1)}(\kappa_1)$$

$$= \frac{4i}{N} V_n^{(2)}(\kappa_1) V_m^{(-1)}(\kappa_1) \tag{7.12}$$

and

$$\oint_{C_{\kappa_0}} d\kappa \mu(N, \kappa) V_n^{(2)}(\kappa) V_m^{(-1)}(\kappa) = 2i \left(\frac{2}{N} - 1\right) V_n^{(2)}(\kappa_0) V_m^{(-1)}(\kappa_0) \tag{7.13}$$

The factor $-2i$ in the r.h.s. of this equation, which is absent in the case $N = 2$, is the contribution of the pole coming from the first factor in the r.h.s. of (7.10), which is the measure appearing in the orthogonality relations. We will forget about this additional factor for the time being and comment about it later on.

Finally

$$T_{(i)nm}^{(N)} = \hat{T}_{nm}^{(N)} - \frac{4}{N} i V_n^{(2)} \left(\frac{4}{N}i\right) V_m^{(-1)} \left(\frac{4}{N}i\right) - \frac{2i}{N} V_n^{(2)}(0) V_m^{(-1)}(0) \tag{7.14}$$

where $V^{(2)}$ and $V^{(-1)}$ stand for the continuous bases evaluated at the corresponding points. Here are some examples of this formula for $N = 4$ (the $N = 3$ coincides with the ghost vertex Neumann coefficients of section 5)

$$(2, -1) : -\frac{3}{2}i = 0 - i - \frac{i}{2}$$

$$(3, 1) : \frac{19}{16} = \frac{11}{16} + \frac{1}{2} + 0$$

$$(4, 2) : -\frac{15}{16} = \frac{5}{16} - \frac{5}{4} + 0$$

$$(5, 4) : \frac{5i}{16} = 0 + \frac{5i}{16} + 0$$

This is precisely what is expected for the BRST invariant dual wedge state specified by the following Neumann coefficients

$$S_{(i)pM}^{(N)} = \oint_0 \frac{dz}{2\pi i} \oint_0 \frac{dw}{2\pi i} \frac{1}{z^{p-1}} \frac{1}{w^{M+2}} \tag{7.15}$$

$$\cdot \left[\frac{f_N'(z)^2}{f_N'(w)} \frac{1}{f_N(z) - f_N(w)} \left(\frac{f_N(w)}{f_N(z)}\right)^3 - \frac{1}{z - w} \right]$$

These are surface states with $Y(i)$ insertion. They are obtained by setting $t = i$ in the appropriate formulas for the Neumann matrix in appendix A.

In appendix D we show how to reconstruct also $T_{(-i)}^{(N)}$.

Now let us make a comment about the factor of $+i$ we disregarded above in the r.h.s. of (7.13). This factor gives rise in the spectral formulas (7.14) to a term proportional to $P_{ni} = V_n^{(2)}(0) V_i^{(-1)}(0)$ (remember that $V_n^{(-1)}(0) = 0$ for $n \geq 2$). Putting everything together, the reconstruction formula for $T_{(i)}^{(N)}$ in terms of contour integration is (see also (5.14)

$$(T_{(i)}^{(N)})_{nm} = \int_{\Im(\kappa) > \frac{4}{N}} d\kappa \, \tilde{V}_n^{(2)}(\kappa) \mathfrak{t}_N(\kappa) \tilde{V}_m^{(-1)}(\kappa) + \sum_\xi \tilde{v}_n^{(2)}(\xi) \mathfrak{t}_N(\xi) \tilde{V}_m^{(-1)}(\xi) - i P_{nm} \quad (7.16)$$

where the contour is a straight line from $-\infty$ to ∞ just above the pole at $\kappa = \frac{4i}{N}$. The rank 1 matrix P commutes with everything else in the spectral formulas and one would be tempted to drop it; however this piece will turn out to be godgiven in paper III.

Let us end with a few important remarks.

Remark 1. In functional analysis the spectral formulas for operators are the sums (integral) of their eigenvalues multiplied by the corresponding eigenprojectors. In (7.14) this corresponds to the first term, $\hat{T}^{(N)}$, in the r.h.s. . The other terms in the r.h.s. are still diagonal and made of eigenprojectors, but the corresponding eigenvalues are infinite and are replaced by the residues of the relevant poles. This is the real novelty of such formulas. We call the former part, the genuinely spectral representation, the *principal part* and the latter *the residual part*. With some abuse of language we will keep referring to formulas like (7.16) as spectral representations, since they are diagonal and contain only information about the spectrum. It is important to notice that all the spectral representations considered in section 5,6 and 7 represent matrices which are completely diagonal in both the continuous and discrete bases of eigenvectors of the matrix G. This characterizes all the ghost number 3 wedge states and marks a sharp difference with the ghost number 0 wedge states, characterized by Neumann matrices which are not completely diagonalizable in the same bases.

Remark 2. The wedge states we have considered in this section are characterized by the fact that they can be represented as squeezed states, but only those with $Y(\pm i)$ insertions are BRST invariant surface states; for the remaining ones the latter properties are open questions and will be rediscussed in III. However we would like to notice that the reconstruction formulas and commutativity of their Neumann matrices hold for all of them.

Remark 3. The spectral formulas are much more effective than the analytic methods from the calculational point of view. In appendix E, where the equation $U^2 = 1$ is proved using the reconstruction formulas, one can find an example of their potential by comparison with the long derivations of section 2.

8 Conclusion

Let us conclude this paper by recalling the main results we have obtained. The first is the construction of the ghost number 9 vertices, eqs. (1.6), (1.9). The second important result is the construction of the discrete bases of eigenvectors of G as well as the bi–orthogonality

and bi–completeness relations (4.15) and (4.17), and the analysis of the highly nontrivial properties of this spectrum. Then we have completed the argument of I, showing that the squeezed states appearing in the midterm of (1.2) do satisfy the recursion relations of the wedge states. The way we have done it is somewhat different from the one envisaged in I. The idea behind I was that all the involved Neumann matrices could be simultaneously and completely diagonalized. In this paper we have realized that this is not possible. Not all the Neumann matrices entering the problem can be completely diagonalized (this will be evident in III). Nevertheless it is still possible to carry out the program started in I. We have shown that the wedge states recursion relations can be proved for the eigenvalues, and that on the basis of this knowledge it is possible to reconstruct ghost number 3 Neumann matrices which can be identified with surface states representing the wedge states expected as a result of the star product. This is enough to guarantee that the three strings vertex we have introduced in section 2 does the job, that is by ∗–multiplying two squeezed states like the ones in the r.h.s. of (1.2) we obtain in the l.h.s. the wedge state required by the recursion relation (1.1). What is still missing is how to recover the the ghost number 0 wedge states from the so obtained ghost number 3 states. This task, which is simply the *bpz* conjugation in the matter sector, requires a very involved and roundabout treatment in the ghost sector and will be dealt with in the next paper.

Acknowledgments

A   Ghost insertions and correlators

The two–point function for a $b - c$ system can take different forms depending on the way we insert the zero modes to soak the background ghost charge at ∞, which is necessary in order to get a nonvanishing result. A generic way is to define, as in [51, 52],

$$\ll c(z)b(w) \gg_{(t_1,t_2,t_3)} = \langle 0|c(z)b(w)c(t_1)c(t_2)c(t_3)|0\rangle$$
$$= \frac{1}{z-w} \prod_{i=1}^{3} \frac{t_i - z}{t_i - w} (t_1 - t_2)(t_1 - t_3)(t_2 - t_3) \qquad (A.1)$$

Another way of inserting the zero modes is by means of the weight 0 operator $Y(t) = \frac{1}{2}\partial^2 c(t)\partial c(t)c(t)$. We have

$$\ll c(z)\,b(w) \gg_t = \langle c(z)\,b(w)\,Y(t)\rangle = \frac{1}{z-w} \frac{(t-z)^3}{(t-w)^3} \qquad (A.2)$$

In this appendix we would like to study the relation, between the normal ordering in the $b - c$ correlator and the ordering term in the matrices of Neumann coefficients of the three

strings vertex. The radial ordering of the b, c fields can be expressed as follows in terms of the natural normal ordering $(::)$:

$$R(c(z)b(w)) = c(z)b(w), \qquad |z| > |w| \tag{A.3}$$

$$= \sum_n c_n z^{-n+1} \left(\sum_{k \leq -2} b_k w^{-k-2} + \sum_{k \geq -1} b_k w^{-k-2} \right)$$

$$= \; : c(z)\, b(w) : + \sum_{n \geq 2} z^{-n+1}\, w^{-k-2}$$

$$= \; : c(z)\, b(w) : + \frac{1}{z - w}$$

The same expression is obtained for $|z| < |w|$.

We can use the above radial ordering in order to get the ordering terms for the Neumann coefficients.

$$R(c(z)b(w)) = \; : c(z)\, b(w) : + \frac{1}{z - w} \longrightarrow \hat{V}^{rs}_{nm} = \ldots - \frac{\delta^{rs}}{z - w}$$

$$\longrightarrow U_{nm} = \ldots - \frac{z}{z - w} \tag{A.4}$$

where only the relevant parts are written down.

Now let us consider a ghost surface state determined by a map $g(z)$,

$$\langle g | = \langle 0 | e^{-\sum c_n S^{(g)}_{nm} b_m} \tag{A.5}$$

In order to find the matrix $S^{(g)}$ we proceed as follows: using (A.2) we identify (see [4, 39]) up to constant factors

$$\langle g | c(z)\, b(w)\, Y(t) \rangle = \frac{(g'(w))^2}{g'(z)} \frac{1}{g(z) - g(w)} \left(\frac{g(t) - g(z)}{g(t) - g(w)} \right)^3 \tag{A.6}$$

with Y insertion at the generic point t. The wedge states are generated by the well-known functions

$$g(z) \equiv g_N(z) = \left(\frac{1 + iz}{1 - iz} \right)^{\frac{2}{N}}$$

If we set the insert Y at $t = 0$ we get

$$S^{(g_N)}_{nm} = \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^{n-1}} \frac{1}{w^{m+2}}$$

$$\cdot \left(\frac{(g'_N(z))^2}{(g'_N(w))} \frac{1}{g_N(z) - g_N(w)} \left(\frac{1 - g_N(w)}{1 - g_N(z)} \right)^3 - \frac{w^3}{z^3(z - w)} \right) \tag{A.7}$$

This is the Neumann matrix for the ghost number 0 wedge states. The others, which represent $gh = 3$ states with a $Y(\pm i)$–insertion, are just obtained by setting $t = \pm i$.

B Quadratic expressions involving $\mathcal{E}$, $\mathcal{U}_{(\pm i)}$ and Z

This appendix is devoted to complete the derivation of the fundamental properties of Neumann coefficients started in section 6.

B.1 Quadratic expressions involving $\mathcal{E}_{(i)}$ and $\mathcal{U}_{(i)}$

The product $\mathcal{E}_{(i)}\mathcal{U}_{(i)}$ can be carried out as in subsection 2.4.1, forgetting the f factors on the first matrix. It is easy to see that it leads to

$$
\begin{aligned}
(\mathcal{E}_{(i)}\mathcal{U}_{(i)})_{nm} &= \oint \frac{dz}{2\pi i}\frac{1}{z^{n+1}}\oint\frac{dw}{2\pi i}\frac{1}{w^{m+1}}\left[-\left(\frac{1}{1-zw}-\frac{w}{z+w}\right)\left(1-p_i(z,-w)\right)\right.\\
&\quad +\frac{z^2 w^2}{1-zw}\\
&\quad \left.+\left(\frac{1}{1-zw}-\frac{w}{z+w}\right)\left(1-p_i(-z,w)\right)\ \frac{f(-z)}{f(w)}\right]\\
&= (-\mathcal{E}_{(i)}\mathcal{C} + \mathbf{1}_{ss} + \mathcal{C}\,\mathcal{U}_{(i)})_{nm}
\end{aligned}
\tag{B.1}
$$

which differs from $\mathcal{C}\mathcal{U}_{(i)}$ only in the zero mode sector. Similarly we can prove that

$$
\mathcal{E}_{(-i)}\mathcal{U}_{(i)} = -\mathcal{E}_{(-i)}\mathcal{C} + \mathbf{1}_{ss} + \mathcal{C}\,\mathcal{U}_{(i)}
\tag{B.2}
$$

Taking the average of these two we obtain

$$
\mathcal{E}\,\mathcal{U}_{(i)} = -\mathcal{E}\mathcal{C} + \mathbf{1}_{ss} + \mathcal{C}\,\mathcal{U}_{(i)}
\tag{B.3}
$$

In a similar way

$$
\mathcal{E}_{(\pm i)}\bar{\mathcal{U}}_{(i)} = -\mathcal{E}_{(\pm i)}\mathcal{C} + \mathbf{1}_{ss} + \mathcal{C}\,\bar{\mathcal{U}}_{(i)}
\tag{B.4}
$$

and

$$
\mathcal{E}_{(\pm i)}\mathcal{E}_{(i)} = -\mathcal{E}_{(\pm i)}\mathcal{C} + \mathbf{1}_{ss} + \mathcal{C}\,\mathcal{E}_{(i)}
\tag{B.5}
$$

Using twist conjugation we finally get

$$
\mathcal{E}\,\mathcal{U} = \mathbf{1}_{ss}
\tag{B.6}
$$

Let us now consider $\mathcal{U}_{(i)}\,\mathcal{E}_{(i)}$.

$$
\sum_{k=-1}^{\infty}\mathcal{U}_{(i)nk}\,\mathcal{E}_{(i)km} = \oint\frac{dz}{2\pi i}\frac{1}{z^{n+1}}\oint\frac{d\zeta}{2\pi i}\oint\frac{d\theta}{2\pi i}\oint\frac{dw}{2\pi i}\frac{1}{w^{m+1}}\frac{\zeta\theta}{\zeta\theta-1}
\tag{B.7}
$$

$$
\cdot\frac{f(z)}{f(\zeta)}\left(\frac{1}{1+z\zeta}-\frac{\zeta}{\zeta-z}\right)\left(1-p_i(z,\zeta)\right)\cdot
$$

$$
\cdot\left(\frac{1}{1+\theta w}-\frac{w}{w-\theta}\right)\left(1-p_i(\theta,w)\right) = *
\tag{B.8}
$$

Here it is more convenient to integrate first with respect to θ. There are two poles at $\theta = \frac{1}{\zeta}, w$ (the poles at $\theta = \pm i$ can be avoided with a regulator). We get

$$\left\{\theta = \frac{1}{\zeta}\right\} \qquad * = \oint \frac{dz}{2\pi i}\frac{1}{z^{n+1}} \oint \frac{d\zeta}{2\pi i} \oint \frac{dw}{2\pi i}\frac{1}{w^{m+1}}\left[\frac{f(z)}{f(\zeta)}\left(\frac{1}{1+z\zeta} - \frac{\zeta}{\zeta - z}\right)\right.$$
$$\cdot \left(1 - p_i(z,\zeta)\right)\left(\frac{\zeta}{\zeta + w} + \frac{\zeta w}{1 - w\zeta}\right)\left(1 - p_i(\tfrac{1}{\zeta}, w)\right)\Big) \qquad \text{(B.9)}$$
$$\{\theta = w\} \qquad + \frac{\zeta w^2}{\zeta w - 1}\frac{f(z)}{f(\zeta)}\left(\frac{1}{1+z\zeta} - \frac{\zeta}{\zeta - z}\right)\left(1 - p_i(z,\zeta)\right)\Big]$$
$$= \oint \frac{dz}{2\pi i}\frac{1}{z^{n+1}} \oint \frac{dw}{2\pi i}\frac{1}{w^{m+1}}\left[\frac{f(z)}{f(-w)}\left(\frac{1}{1 - zw} - \frac{w}{w + z}\right)\right.$$
$$\cdot\left(1 - p_i(z,-w)\right) + \frac{w^2 z^2}{1 - zw}$$
$$- \left(\frac{w}{z + w} - \frac{1}{1 - zw}\right)\left(1 - p_i(-z,w)\right)\Big]$$
$$= (\mathcal{U}_{(i)}\,\mathcal{C} - \mathcal{C}\,\mathcal{E}_{(i)} + \mathbf{1}_{ss})_{nm}$$

Similarly

$$\mathcal{U}_{(-i)}\mathcal{E}(i) = \mathcal{U}_{(-i)}\,\mathcal{C} - \mathcal{C}\,\mathcal{E}_{(i)} + \mathbf{1}_{ss}$$
$$\bar{\mathcal{U}}_{(i)}\mathcal{E}(i) = \bar{\mathcal{U}}_{(i)}\,\mathcal{C} - \mathcal{C}\,\mathcal{E}_{(i)} + \mathbf{1}_{ss} \qquad \text{(B.10)}$$
$$\bar{\mathcal{U}}_{(-i)}\mathcal{E}(i) = \bar{\mathcal{U}}_{(-i)}\,\mathcal{C} - \mathcal{C}\,\mathcal{E}_{(i)} + \mathbf{1}_{ss}$$

from which it is easy to deduce the quadratic relations involving $\mathcal{E}$.

B.2 Quadratic expressions involving Z

Let us now compute

$$\sum_{k=-1}^{\infty} Z_{nk}\,\mathcal{U}_{(\pm i)km} = \oint \frac{dz}{2\pi i}\frac{1}{z^{n+1}} \oint \frac{d\zeta}{2\pi i} \oint \frac{d\theta}{2\pi i} \oint \frac{dw}{2\pi i}\frac{1}{w^{m+1}}\frac{\zeta\theta}{\zeta\theta - 1} \qquad \text{(B.11)}$$
$$\cdot\left(-\frac{z^2}{\zeta}\frac{1}{z - w}\right)\frac{f(\theta)}{f(w)}\left(\frac{1}{1+\theta w} - \frac{w}{w - \theta}\right)\left(1 - p_{\pm i}(\theta, w)\right) = * \quad \text{(B.12)}$$

Now we proceed as above and the result is

$$\left\{\zeta = \frac{1}{\theta}\right\} \qquad * = \oint \frac{dz}{2\pi i}\frac{1}{z^{n+1}} \oint \frac{d\zeta}{2\pi i} \oint \frac{dw}{2\pi i}\frac{1}{w^{m+1}}\left[\frac{\theta z^2}{1 - z\theta}\frac{f(\theta)}{f(w)}\left(\frac{1}{1+\theta w} - \frac{w}{w - \theta}\right)\cdot\right.$$
$$\cdot\left(1 - p_{\pm i}(\theta, w)\right) \qquad \text{(B.13)}$$
$$\{\zeta = z\} \qquad + \frac{\theta z^2}{\theta z - 1}\frac{f(\theta)}{f(w)}\left(\frac{1}{1+\theta w} - \frac{w}{w - \theta}\right)$$
$$\cdot\left(1 - p_{\pm i}(\theta, w)\right)\Big] = 0$$

By twist conjugation $Z\bar{\mathcal{U}}_{\pm i} = 0$.

The calculation of $Z\mathcal{E}_{\pm i}$ is similar but simpler (no f factor) and the conclusion is the same: $Z\mathcal{E}_{\pm i} = 0$.

Consider now $\mathcal{U}_{(i)} Z$:

$$\sum_{k=-1}^{\infty} \mathcal{U}_{(i)nk} Z_{km} = \oint \frac{dz}{2\pi i} \frac{1}{z^{n+1}} \oint \frac{d\zeta}{2\pi i} \oint \frac{d\theta}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{w^{m+1}} \frac{\zeta\theta}{\zeta\theta - 1} \qquad \text{(B.14)}$$
$$\cdot \frac{f(z)}{f(\zeta)} \Big(\frac{1}{1+z\zeta} - \frac{\zeta}{\zeta - z}\Big) \Big(1 - p_i(z,\zeta)\Big) \cdot$$
$$\cdot \Big(-\frac{\theta^2}{w} \frac{1}{\theta - w}\Big) = *$$

Here it is more convenient to integrate first with respect to θ. There are two poles at $\theta = \frac{1}{\zeta}, w$.

$$\left\{\theta = \frac{1}{\zeta}\right\} \qquad * = \oint \frac{dz}{2\pi i} \frac{1}{z^{n+1}} \oint \frac{d\zeta}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{w^{m+1}} \left[\frac{f(z)}{f(\zeta)}\right.$$
$$\cdot \Big(\frac{1}{1+z\zeta} - \frac{\zeta}{\zeta - z}\Big) \Big(1 - p_i(z,\zeta)\Big) \Big(-\frac{1}{w\zeta^2} \frac{1}{1 - \zeta w}\Big) \qquad \text{(B.15)}$$
$$\{\theta = w\} \qquad -\frac{\zeta w^2}{\zeta w - 1} \frac{f(z)}{f(\zeta)} \Big(\frac{1}{1+z\zeta} - \frac{\zeta}{\zeta - z}\Big) \Big(1 - p_i(z,\zeta)\Big)\right]$$
$$= \oint \frac{dz}{2\pi i} \frac{1}{z^{n+1}} \oint \frac{d\zeta}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{w^{m+1}} \frac{f(z)}{f(\zeta)}$$
$$\cdot \Big(\frac{1}{1+z\zeta} - \frac{\zeta}{\zeta - z}\Big) \Big(1 - p_i(z,\zeta)\Big) \Big(-\frac{1}{w\zeta^2}(1 + w\zeta + w^2\zeta^2)\Big) = **$$

This gives

$$** = \oint \frac{dz}{2\pi i} \frac{1}{z^{n+1}} \oint \frac{dw}{2\pi i} \frac{1}{w^{m+1}} \left\{\frac{1}{2} \frac{d^2}{d\zeta^2} \left[\zeta \frac{f(z)}{f(\zeta)}\right.\right.$$
$$\cdot \Big(\frac{1}{1+z\zeta} - \frac{\zeta}{\zeta - z}\Big) \Big(1 - p_i(z,\zeta)\Big) \Big(-\frac{1}{w}(1 + w\zeta + w^2\zeta^2)\Big)\Big]_{\zeta=0}$$
$$\left. + \frac{1}{wz}(1 + wz + w^2 z^2)\right\}$$
$$= \oint \frac{dz}{2\pi i} \frac{1}{z^{n+1}} \oint \frac{dw}{2\pi i} \frac{1}{w^{m+1}} \left(\frac{f(z)}{zw} \frac{P_i(z,w)}{9(z-i)^2} + \frac{1}{wz} + 1 + wz\right) \qquad \text{(B.16)}$$

where

$$P_i(z,w) = 9 + 7z^2 + 9z^4 + 6iz(1 - z^2) + 3wz(3 - 3z^2 + 2iz) + 9w^2 z^2 \qquad \text{(B.17)}$$

$P_i(z,w)$ is a polynomial of fourth order in z and second order in w. Therefore the $\mathcal{U}_{(i)} Z$ matrix vanishes except for the first three columns.

The result for $\bar{\mathcal{U}}_{(i)} Z$ is the same with the substitution $P_i \to P_{-i}$,

$$P_{-i}(z,w) = 9 + 30iz - 41z^2 - 30iz^2 + 9z^4 + w(9z + 30iz^2 - 9z^3) + 9w^2 z^2 \qquad \text{(B.18)}$$

The calculation for $\mathcal{E}_{(i)} Z$ is similar but simpler and the conclusion is

$$(\mathcal{E}_{(i)}Z)_{nm} = \oint \frac{dz}{2\pi i} \frac{1}{z^{n+1}} \oint \frac{dw}{2\pi i} \frac{1}{w^{m+1}} \left(\frac{1}{zw} \frac{Q_i(z,w)}{(z-i)^2} + \frac{1}{wz} + 1 + wz \right) \qquad (B.19)$$

where

$$Q_i(z,w) = 1 + 2iz - z^2 - 2iz^3 + z^4 + w(z + 2iz^2 - z^3) + w^2 z^2 \qquad (B.20)$$

$Q_i(z,w)$ is a polynomial of fourth order in z and second order in w. The $\mathcal{E}_{(i)} Z$ matrix vanishes except for the first three columns. By twist conjugation one gets $\mathcal{E}_{(-i)} Z$.

It is easy to see that $Z^2 = 0$.

In the sequel we will need more explicit expressions for the r.h.s. of (B.16) and (B.19). To this end let us compute $U_{(i)n,i}$ for $-1 \leq i \leq 1$ in a more explicit form

$$\begin{aligned} U_{(i)n,-1} &= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^{n+1}} \left(\frac{f(z)}{f(w)} \Big(\frac{1}{1+zw} - \frac{w}{w-z} \Big) (1 - p_i(z,w)) - \frac{z^2}{w} \frac{1}{z-w} \right) \\ &= \oint \frac{dz}{2\pi i} \frac{1}{z^{n+1}} \left(-\frac{z f(z)}{(z-i)^2} - z \right) \end{aligned} \qquad (B.21)$$

In this calculation only the pole at $w = 0$ matters, while the pole at $w = z$ has a vanishing residue. Similarly

$$\begin{aligned} U_{(i)n,0} &= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^{n+1}} \frac{1}{w} \left(\frac{f(z)}{f(w)} \Big(\frac{1}{1+zw} - \frac{w}{w-z} \Big) (1 - p_i(z,w)) - \frac{z^2}{w} \frac{1}{z-w} \right) \\ &= \oint \frac{dz}{2\pi i} \frac{f(z)}{z^{n+1}} \frac{d}{dw} \left(w f(-w) \Big(\frac{1}{1+zw} - \frac{w}{w-z} \Big) (1 - p_i(z,w)) \right)_{w=0} \\ &= \oint \frac{dz}{2\pi i} \frac{1}{z^{n+1}} \left(f(z) \frac{-3 - 2iz + 3z^2}{3(z-i)^2} - 1 \right) \end{aligned} \qquad (B.22)$$

Finally

$$\begin{aligned} U_{(i)n,1} &= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^{n+1}} \frac{1}{w^2} \left(\frac{f(z)}{f(w)} \Big(\frac{1}{1+zw} - \frac{w}{w-z} \Big) (1 - p_i(z,w)) - \frac{z^2}{w} \frac{1}{z-w} \right) \\ &= \oint \frac{dz}{2\pi i} \frac{f(z)}{z^{n+1}} \left[\frac{1}{2} \frac{d^2}{dw^2} \left(w f(-w) \Big(\frac{1}{1+zw} - \frac{w}{w-z} \Big) (1 - p_i(z,w)) \right)_{w=0} - \frac{1}{z} \right] \\ &= \oint \frac{dz}{2\pi i} \frac{1}{z^{n+1}} \left(-f(z) \frac{9 + 7z^2 + 9z^4 + 6iz(1-z^2)}{9z(z-i)^2} - \frac{1}{z} \right) \end{aligned} \qquad (B.23)$$

Comparing these expressions with (B.16) and (B.17) it is evident that

$$(\mathcal{U}_{(i)}Z)_{n,i} = -U_{(i)n,-i}, \quad \text{i.e.} \quad (U_{(i)}Z)_{n,i} = -U_{n,-i} \qquad (B.24)$$

As for $\bar{\mathcal{U}}_{(i)}$, we have

$$\bar{U}_{(i)n,-1} = \oint \frac{dz}{2\pi i} \frac{1}{z^{n+1}} \left(-\frac{z f(-z)}{(z-i)^2} - z \right) \qquad (B.25)$$

$$\bar{U}_{(i)n,0} = \oint \frac{dz}{2\pi i} \frac{1}{z^{n+1}} \left(f(-z) \frac{-9 - 30iz + 9z^2}{9(z-i)^2} - 1 \right) \qquad (B.26)$$

$$\bar{U}_{(i)n,1} = \oint \frac{dz}{2\pi i} \frac{1}{z^{n+1}} \left(f(-z) \frac{-9 - 30iz + 41z^2 + 30iz^3 - 9z^4}{9z(z-i)^2} - \frac{1}{z} \right) \qquad (B.27)$$

Therefore

$$(\bar{\mathcal{U}}_{(i)} Z)_{n,i} = -\bar{U}_{(i)n,-i}, \quad \text{i.e.} \quad (\bar{U}_{(i)} Z)_{n,i} = -\bar{U}_{n,-i} \tag{B.28}$$

On the other hand it is even easier to prove that

$$\begin{aligned}
E_{(i)n,-1} &= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^{n+1}} \left[\left(\frac{1}{1+zw} - \frac{w}{w-z} \right) (1 - p_i(z,w)) - \frac{z^2}{w} \frac{1}{z-w} \right] \\
&= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^{n+1}} \left(-\frac{z}{(z-i)^2} - z \right) = E_{(i)n,1} \tag{B.29}
\end{aligned}$$
$$E_{(i)n,0} = 0$$

Comparing with the r.h.s. of (B.19) and the expression for $Q_i(z,w)$ we can conclude that

$$(\mathcal{E}_{(i)} Z)_{n,1} = (\mathcal{E}_{(i)} Z)_{n,-1} = -E_{(i)n,1} = -E_{(i)n,-1}, \qquad (\mathcal{E}_{(i)} Z)_{n,0} = 0 \tag{B.30}$$

i.e.

$$(E_{(i)} Z)_{n,1} = (E_{(i)} Z)_{n,-1} = -E_{(i)n,1} = -E_{(i)n,-1}, \qquad (E_{(i)} Z)_{n,0} = 0 \tag{B.31}$$

Analogously one can show that

$$(E\, Z)_{n,1} = (E\, Z)_{n,-1} = -E_{n,1} = -E_{n,-1}, \qquad (E\, Z)_{n,0} = 0 \tag{B.32}$$

For future use we record also

$$\begin{aligned}
E_{n,0} &= 0 \\
E_{2n,1} &= E_{2n,-1} = 0 \\
E_{2n+1,1} &= E_{2n+1,-1} = (-1)^n (2n+1)
\end{aligned} \tag{B.33}$$

C The eigenvalues of T_n

The explicit form of the squeezed states in the midterm of (1.2) was derived from the l.h.s. of the same equation in I. It required solving the equation we dubbed KP, [48]. The latter arises from writing

$$e^{t\left(\mathcal{L}_0^{(g)} + \mathcal{L}_0^{(g)\dagger}\right)} \equiv e^{t\left(c_M^\dagger A_{Mn} b_n^\dagger + c_M^\dagger C_{MN} b_N + b_m^\dagger D_{mn} c_n - c_m B_{mN} b_N\right)} \tag{C.1}$$

where $t = \frac{2-n}{2}$, and equating this to

$$e^{\eta(t)} e^{c^\dagger \alpha(t) b^\dagger} e^{c^\dagger \gamma(t) b} e^{b^\dagger \delta(t) c} e^{c\beta(t) b} \tag{C.2}$$

The matrix α and the parameter η, in particular, must satisfy

$$\dot{\alpha} = A + C\,\alpha + \alpha\,D^T + \alpha\,B\,\alpha \tag{C.3}$$

and

$$\dot{\eta} = -\text{Tr}\,(B,\alpha) \tag{C.4}$$

In I we solved this equation. However, for reasons explained later on, we will proceed here in a different way. Instead of solving (C.3) and then diagonalizing the solution, we will diagonalize (C.3) and then solve the equation for the eigenvalues. Therefore our first task is to find the eigenvalues of the various matrices appearing in (C.3). In I we showed that $\tilde{A}, D^T$ and $(D^T)^2 - BA$ are diagonal in the continuous weight 2 basis and computed the explicit eigenvalues. The analysis below will confirm this.

C.1 Revisiting I

This subsection is devoted to re–deriving the solution to the KP equation with respect to I. The reason for it is the following. In I we used the following 'commutation rules' for lame matrices:

$$AD^T = CA \tag{C.5}$$

and

$$BC = D^T B \tag{C.6}$$

The latter has to be qualified. It is true except for the terms

$$(BC - D^T B)_{2,0} = -\frac{3}{2}\pi^2, \qquad (BC - D^T B)_{3,-1} = -3\pi^2 \tag{C.7}$$

Therefore in the sequel, differently from I, we will not use eq. (C.6). We will only use (C.5). But, of course, we have to change the strategy with respect to I. Instead of solving (C.3) and then diagonalizing it, we will first diagonalize (C.3) and then solve for its eigenvalues. Let us start from eq. (C.3) with the initial condition $\alpha(0) = 0$. The solution to (C.4) is obvious once we know $\alpha(t)$. Let us make the following ansatz for $\alpha(t)$

$$\alpha_1(t) = A\,Q_1(t) \tag{C.8}$$

Using (C.5) we get

$$A\,\dot{Q}_1 = A\,(1 + D^T Q_1 + Q_1 D^T + Q_1 BA\,Q_1) \tag{C.9}$$

It is obvious that, if Q_1 satisfies

$$\dot{Q}_1 = 1 + D^T Q_1 + Q_1 D^T + Q_1 BA\,Q_1 \tag{C.10}$$

with $Q_1(0) = 0$, α_1 will satisfy (C.3).

Next we wish to solve (C.10) for the continuous eigenvalues of the matrix $Q_1(t)$. We have recalled above that the matrix D^T is diagonal in the $V^{(2)}(\kappa)$ basis, with eigenvalue $\mathfrak{c}(\kappa)$ and that $(D^T)^2 - BA$ is also diagonal. This means that BA itself is diagonal in the same basis. Looking at (C.10) we see that since the solution $Q(t)$ will be a function of D^T and BA it will also be diagonal in the same basis. So in (C.10) we can replace the matrices with their eigenvalues. At this point solving the equation is elementary.

$$Q_1(t) = \frac{\sinh\left(\sqrt{(D^T)^2 - BA}\,t\right)}{\sqrt{(D^T)^2 - BA}\cosh\left(\sqrt{(D^T)^2 - BA}\,t\right) - D^T \sinh\left(\sqrt{(D^T)^2 - BA}\,t\right)} \tag{C.11}$$

where, for economy of notation, we assume that the matrices represent their continuous eigenvalues. Since A is also diagonal on the $V^{(2)}(\kappa)$ basis, we can mutiply this solution by its eigenvalue and get

$$\alpha_1(t) = A\, Q(t) \tag{C.12}$$

for the corresponding continuous eigenvalues. Now from the continuous eigenvalues of A, BA and D^T we can construct the continuous eigenvalue of $\alpha_1(t)$. This has already been done in I, and the result, $\mathfrak{t}_t(\kappa)$, is that reported in eq. (5.3).

C.2 The discrete eigenvalues

Coming now to the discrete eigenvalues we would like to be able to write

$$V^{(-1)}(\xi)\tilde{\alpha}(t) = \mathfrak{t}_t(\xi)V^{(-1)}(\xi) \tag{C.13}$$

or

$$\tilde{\alpha}(t)\, V^{(2)}(\xi) = \mathfrak{t}_t(\xi)\, V^{(2)}(\xi), \tag{C.14}$$

calculate the explicit expression of $\mathfrak{t}_n(\xi)$ and justify our final statements $\mathfrak{t}_n(\pm 2i) = 1$ and $\mathfrak{t}_n(\xi = 0) = -1$, eqs. (C.33) and (C.34). But if we apply the previous matrices to the discrete eigenvectors of G we immediately run into difficulties. If we use the same ansatz (C.8) and eq. (C.10) we see that, for instance, D^T has identically vanishing eigenvalues in the $V^{(-1)}(\xi)$ basis. This is a consequence of the singular nature of the discrete eigenvalues of the $\tilde{A}, \tilde{B}, C, D$ matrices, which creates serious problems when we try to compute the corresponding discrete eigenvalues of the wedge states by integrating the KP equation. These problems will be understood in the next paper, when we will be able to produce the reconstruction formula for the ghost number 0 wedge states. For the time being we proceed blindly in search of these eigenvalues.

As a preliminary step let us apply C from the right to the discrete basis $V^{(-1)}(\xi) = (V^{(-1)}_{-1}(\xi), V^{(-1)}_0(\xi), \ldots)$. We get, for $\xi = 0$,

$$(V^{(-1)}(0)\, C)_n = C_{1,n} + C_{-1,n} = -A_{-1,n} - A_{1,n} = 0 \tag{C.15}$$

while for $\xi = \pm 2i$,

$$\begin{aligned}(V^{(-1)}(\pm 2i)\, C)_n &= C_{1,n} - C_{-1,n} \pm 2iC_{0,n} = -A_{-1,n} + A_{1,n} \mp 2iA_{0,n}\\ &= \lim_{x\to 0}\left(\pm\frac{2i}{x}\, V^{(-1)}_n(\pm 2i + x)\right)\end{aligned} \tag{C.16}$$

for $n \geq 2$. Therefore we can write

$$(V^{(-1)}(\xi)\, C)_n = \lim_{x\to 0}\left(\frac{\xi}{x}\, V^{(-1)}_n(\xi + x)\right) \tag{C.17}$$

but

$$(V^{(-1)}(\xi)\, C)_j = \xi^2 V^{(-1)}(\xi = 0) \tag{C.18}$$

In summary we have the eigenvalue equation

$$(V^{(-1)}(\xi)\,C)_N = \lim_{x \to 0} \left(\frac{\xi}{x} V_N^{(-1)}(\xi + x) \right), \quad N \geq -1 \tag{C.19}$$

for $\xi = 0$, but we have to get along with (C.17), (C.18) for $\xi \neq 0$. To give a meaning to (C.19) we agree that the eigenvalue $\xi = 0$ will be represented by a small number ϵ, and that the latter will be sent to 0 at the end of the calculations.

Likewise we get

$$(V^{(-1)}(0)\,A)_n = A_{1,n} + A_{-1,n} = 0 \tag{C.20}$$

and

$$(V^{(-1)}(\pm 2i)\,A)_n = A_{1,n} - A_{-1,n} \pm 2iA_{0,n} = \lim_{x \to 0} \left(\mp \frac{2i}{x} V_n^{(-1)}(\pm 2i + x) \right) \tag{C.21}$$

Summarizing

$$(V^{(-1)}(\pm 2i)\,A)_n = \lim_{x \to 0} \left(\mp \frac{2i}{x} V_n^{(-1)}(\pm 2i + x) \right), \qquad n \geq 2 \tag{C.22}$$

$$(V^{(-1)}(0)\,A)_n = \lim_{x \to 0, \xi \to 0} \left(\frac{\xi}{x} V_n^{(-1)}(\xi + x) \right) = 0, \quad n \geq 2 \tag{C.23}$$

$$(V^{(-1)}(\xi)\,A)_i = 0, \qquad\qquad\qquad -1 \leq i \leq 1 \tag{C.24}$$

In (C.23) the limits have to be taken in such a way that the result be 0, so, as above, we introduce a regulator ϵ to represent the eigenvalue $\xi = 0$. Notice that $\lim_{x \to 0} V_n^{(-1)}(\xi + x) = 0$ for $n \geq 2$, while $\lim_{x \to 0} V_i^{(-1)}(\xi + x) = V_i^{(-1)}(\xi)$ and that $V_n^{(-1)}(\xi + x)$ for $n \geq -1$ is bi-orthogonal to both $V_n^{(2)}(\xi)$ and $V_n^{(2)}(\kappa)$ in the limit $x \to 0$.

Apart from the eigenvalue equation (C.19), the above equations (C.17), (C.18), (C.22), (C.23), (C.24) represent 'almost' eigenvalue equations. They will be used to compute the discrete eigenvalues of the wedge state Neumann matrices.

In I we proved, eq. (5.19), that $\Delta = C^2 - AB$ is diagonal in the continuous basis $V^{(-1)}(\kappa)$. It is easy to prove that it is diagonal also in the discrete basis, using (4.21) of I. For instance it is elementary to prove that

$$\sum_{n=-1} V_n^{(-1)}(0)\Delta_{n,0} = 0$$

$$\sum_{n=-1} V_n^{(-1)}(0)\Delta_{n,\pm 1} = 0$$

$$\sum_{n=-1} V_n^{(-1)}(2i)\Delta_{n,\pm 1} = -\pi^2 V_{\pm 1}^{(-1)}(2i)$$

$$\sum_{n=-1} V_n^{(-1)}(2i)\Delta_{n,0} = -\pi^2 V_0^{(-1)}(2i)$$

So we can write

$$\sum_{n=-1} V_n^{(-1)}(\xi)\Delta_{n,m} = \frac{\pi^2}{4}\xi^2\, V_m^{(-1)}(\xi) \tag{C.25}$$

Thanks to these results, we set out to compute the discrete eigenvalues of the wedge state Neumann matrices.

We make a new ansatz for the solution to the KP equation

$$\alpha_2(t) = Q_2(t)A \tag{C.26}$$

Using (C.5) we get

$$\dot{Q}_2 A = (1 + Q_2 C + C Q_2 + Q_2 AB Q_2)A \tag{C.27}$$

It is obvious that, if Q_2 satisfies

$$\dot{Q}_2 = 1 + Q_2 C + C Q_2 + Q_2 AB Q_2 \tag{C.28}$$

with $Q_2(0) = 0$, α_2 will satisfy (C.3). Now, let us remember that C and $C^2 - AB$ are diagonal in the $V^{(-1)}(\xi)$ basis as far as the $\xi = \epsilon \to 0$ eigenvalue is concerned. Arguing as before we can conclude that also AB is diagonal. Proceeding from now on for this single eigenvalue, we can replace the matrices in (C.28) with their discrete eigenvalues. The solution to (C.28) is

$$Q_2(t) = \frac{\sinh\left(\sqrt{C^2 - AB}\, t\right)}{\sqrt{C^2 - AB}\cosh\left(\sqrt{C^2 - AB}\, t\right) - C\sinh\left(\sqrt{C^2 - AB}\, t\right)} \tag{C.29}$$

where the matrix symbols have to be understood as representing the corresponding discrete eigenvalues. Now remember that A is also diagonal in the same basis. Therefore

$$\alpha_2(t) = \frac{\sinh\left(\sqrt{C^2 - AB}\, t\right)}{\sqrt{C^2 - AB}\cosh\left(\sqrt{C^2 - AB}\, t\right) - C\sinh\left(\sqrt{C^2 - AB}\, t\right)} A \tag{C.30}$$

The multiplication by (the eigenvalue of) A requires a specification. In fact applying the equation (C.27) from the right to $V^{(-1)}$, everything is all right as far as the last A factor on the right. When applying this to $V^{(-1)}$ the first three entries on the r.h.s. of (C.27) get cut out.

At this point however we can apply a remark similar to the one made in section 2.4. An expression like $e^{c_N^\dagger \tilde{\alpha}_{Nm}(t) b_m^\dagger}$ in (1.2) manifests an ambiguity when applied to $|0\rangle$. We could add any term that is killed by $|0\rangle$. This is the case if we consider $:\, e^{c_i^\dagger \tilde{\tau}_{ij} b_j^\dagger + c_N^\dagger \tilde{\alpha}_{Nm}(t) b_m^\dagger}\,:$ $|0\rangle$, for

$$:\, e^{c_i^\dagger \tau_{ij} b_j^\dagger + c_N^\dagger \tilde{\alpha}_{Nm}(t) b_m^\dagger}\,: |0\rangle = e^{c_N^\dagger \tilde{\alpha}_{Nm}(t) b_m^\dagger}|0\rangle$$

for any 3×3 matrix τ. We therefore take advantage of this ambiguity by adding to $\alpha_2(t)$ an upper left 3×3 non zero matrix that solves the problem. The latter is constructed as follows. Let us denote by $\mathfrak{t}_t(\xi)$ the discrete eigenvalues of $\tilde{\alpha}_2(t)$, then the 3×3 matrix we are looking for will be

$$\tilde{\alpha}_{3\times 3}(t) = \sum_\xi \tilde{\tilde{V}}^{(2)}(\xi)\, \mathfrak{t}_t(\xi)\, \tilde{V}^{(-1)}(\xi) \tag{C.31}$$

where $\bar{V}^{(2)}(\xi)$ is $V^{(2)}(\xi)$ limited to the first three entries. By adding this matrix to $\alpha_2(t)$ now we recreate the missing entries in the r.h.s. of (C.13). Setting $\alpha_2'(t) = \alpha_2(t) + \alpha_{3\times3}(t)$ we can now write

$$V^{(-1)}\tilde{\alpha}_2'(t) = \mathfrak{t}_t(\xi)V^{(-1)} \tag{C.32}$$

We are now in the condition to compute the discrete eigenvalue of $\alpha_2(t)$ for $\xi = 0$. We insert the corresponding discrete eigenvalue of A, C, Δ calculated above. eq. (C.30) become rather singular and some care has to be used: one must take $x \to 0$ with $\xi = \epsilon$ first. As one can see the eigenvalue of A and C explode and the first term in the denominator of (C.30) becomes irrelevant. Since A and C have the same eigenvalue the result is

$$\mathfrak{t}_n(\xi = 0) = -1 \tag{C.33}$$

We notice that this result is not affected by the limit $\epsilon \to 0$.

As for $\xi \neq 0$, try as we may, we cannot repeat the same derivation. The reason for these failures will be understood in the next paper, when it will become clear that the Neumann matrices for ghost number 0 wedge states are not diagonal in the $V^{(-1)}(\xi)$ bases with $\xi \neq 0$. However these eigenvalues are important for the ghost number 3 states. Since the continuous eigenvalue formula (5.3) evaluated at $\xi = \pm 2i$ gives an unambiguous result (keep t generic and use standard trigonometric identities):

$$\mathfrak{t}_n(\pm 2i) = 1 \tag{C.34}$$

it logical to try this. As we have shown in section 5,6 and 7 this turns out to be the correct value.

Finally, let us remark that, inserting these values in (C.31), we find once again the matrix

$$\tilde{\alpha}_{3\times3}(n) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \tag{C.35}$$

Let us also remark an important feature of the addition of the matrix z. The way we implement it is by adjoining it to the matrix A, i.e. $A \to A' = A + z$. Looking at eq. (C.27) we notice that $A'B = AB$, therefore the matrix z disappears from the equation for Q_2. It appears only in the last step when we reconstruct the solution $\alpha_2 = Q_2 A'$.

D   Other reconstructions

D.1   Reconstruction of $X_{(i)}^{\pm}$

In this section, we deal with the reconstruction of the matrices $X_{(i)}^{+} = \mathcal{C}\hat{V}_{(i)}^{12}$ and $X_{(i)}^{-} = \mathcal{C}\hat{V}_{(i)}^{21}$. The process of reconstructing these matrices runs analogously to the one of reconstructing $X_{(i)}$, just with some slight differences. The first of them is that, in this case,

the discrete spectrum does not contribute, hence we have just to consider the continuous part of the spectrum

$$X^{\pm}_{(i)nm} = \int_{C_1} d\kappa \, \mu^{\pm}(\kappa) V_n^{(2)}(\kappa) V_m^{(-1)}(\kappa), \quad \mu^{\pm}(\kappa) = -(1 + e^{\pm \frac{\pi\kappa}{2}})\mu(\kappa) \qquad \text{(D.1)}$$

where C_1 is a straight contour from $-\infty$ to $+\infty$ with $\frac{4}{3} < \Im(\kappa) < 2$. Note that for off–diagonal Neumann coefficients there is no need to correct the contour shifting with the universal nilpotent P_{nm} discussed in section 7.1. If we move the upper contour toward the real axis we are bound to meet two poles of the measure, one at $\kappa = \kappa_1 = \frac{4i}{3}$ and another at $\kappa = \kappa_0 = 0$. So finally we obtain the usual integral along the real axis (which corresponds to $X^{\pm}$) plus two contributions from the two poles that remain trapped inside the contour. The latter are clockwise oriented, so we have to change the sign when calculating the residues. We have then

$$\begin{aligned}
X^{\pm}_{(i)nm} = X^{\pm}_{nm} &- \oint_{\kappa_1 = \frac{4i}{3}} d\kappa \, \mu^{\pm}(\kappa) V_n^{(2)}(\kappa) V_m^{(-1)}(\kappa) \\
&- \frac{1}{2} \oint_{\kappa_0 = 0} d\kappa \, \mu^{\pm}(\kappa) V_n^{(2)}(\kappa) V_m^{(-1)}(\kappa)
\end{aligned} \qquad \text{(D.2)}$$

This gives us finally

$$X^{\pm}_{(i)nm} = X^{\pm}_{nm} - \frac{2i}{3} V_n^{(2)}(0) V_m^{(-1)}(0) + \frac{2i}{3}(1 \pm i\sqrt{3}) V_n^{(2)}\left(\frac{4i}{3}\right) V_m^{(-1)}\left(\frac{4i}{3}\right) \qquad \text{(D.3)}$$

Now, let us see some examples:

$$\begin{aligned}
X^+_{(i)2,0} &= \frac{16}{27} - 0 - \frac{8}{9}(1 + i\sqrt{3}) = -\frac{8}{27}(1 + 3i\sqrt{3}) \\
X^+_{(i)2,-1} &= \frac{2}{3\sqrt{3}} - \frac{2i}{3} - \frac{2}{3}(-i + \sqrt{3}) = -\frac{4}{3\sqrt{3}} \\
X^+_{(i)3,0} &= \frac{64}{81\sqrt{3}} - 0 + \frac{32}{27}(-i + \sqrt{3}) = \frac{32}{243}(-9i + 11\sqrt{3}) \\
X^+_{(i)3,3} &= \frac{10112}{19683} - 0 + \frac{320}{2187}(1 + i\sqrt{3}) = \frac{64}{19683}(203 + 45i\sqrt{3})
\end{aligned}$$

and

$$\begin{aligned}
X^-_{(i)2,0} &= \frac{16}{27} - 0 + \frac{8}{9}i(i + \sqrt{3}) = \frac{8}{27}(-1 + 3i\sqrt{3}) \\
X^-_{(i)2,-1} &= -\frac{2}{3\sqrt{3}} - \frac{2i}{3} + \frac{2}{3}(i + \sqrt{3}) = \frac{4}{3\sqrt{3}} \\
X^-_{(i)3,0} &= -\frac{64}{81\sqrt{3}} - 0 - \frac{32}{27}(i + \sqrt{3}) = -\frac{32}{243}(9i + 11\sqrt{3}) \\
X^-_{(i)3,3} &= \frac{10112}{19683} - 0 + \frac{320}{2187}(1 - i\sqrt{3}) = \frac{64}{19683}(203 - 45i\sqrt{3})
\end{aligned}$$

These results agree perfectly with the ones computed directly using (1.6) (multiplied by $\mathcal{C}$ to give the corresponding X's).

D.2 Reconstruction of $X_{(-i)}^{\pm}$

In order to reconstruct $X_{(-i)}^{+} = \mathcal{C}\hat{V}_{(-i)}^{12}$ and $X_{(-i)}^{-} = \mathcal{C}\hat{V}_{(-i)}^{21}$, it is clear that we should use a different prescription than the one used above. This prescription can be inferred from the fact that $X_{(-i)}^{+} = \mathcal{C}X_{(i)}^{-}\mathcal{C}$ and $X_{(-i)}^{-} = \mathcal{C}X_{(i)}^{+}\mathcal{C}$. We should use then

$$X_{(-i)nm}^{\pm} = \int_{C_2} d\kappa \; \mu^{\mp}(\kappa) V_n^{(2)}(\kappa) V_m^{(-1)}(\kappa), \quad \mu^{\mp}(\kappa) = -(1 + e^{\mp \frac{\pi\kappa}{2}})\mu(\kappa) \qquad (D.4)$$

where C_2 is a straight contour from $-\infty$ to $+\infty$ with $-2 < \Im(\kappa) < -\frac{4i}{3}$. If we move the lower contour toward the real axis we are bound to meet two poles of the measure, one at $\kappa = \kappa_1 = -\frac{4i}{3}$ and another at $\kappa = \kappa_0 = 0$. So finally we obtain the usual integral along the real axis (which corresponds to $X^{\pm}$) plus two contributions from the two poles that remain trapped inside the contour. The latter are anticlockwise oriented this time, so we do not have to change the sign when calculating the residues. We then have

$$X_{(-i)nm}^{\pm} = X_{nm}^{\pm} + \oint_{\kappa_1 = -\frac{4i}{3}} d\kappa \; \mu^{\mp}(\kappa) V_n^{(2)}(\kappa) V_m^{(-1)}(\kappa)$$
$$+ \frac{1}{2} \oint_{\kappa_0 = 0} d\kappa \; \mu^{\mp}(\kappa) V_n^{(2)}(\kappa) V_m^{(-1)}(\kappa) \qquad (D.5)$$

This gives us finally

$$X_{(-i)nm}^{\pm} = X_{nm}^{\pm} + \frac{2i}{3} V_n^{(2)}(0) V_m^{(-1)}(0) - \frac{2i}{3}(1 \mp i\sqrt{3}) V_n^{(2)}\left(-\frac{4i}{3}\right) V_m^{(-1)}\left(-\frac{4i}{3}\right) \qquad (D.6)$$

Let us check some components:

$$X_{(-i)2,0}^{+} = \frac{16}{27} + 0 + \frac{8}{9}i(i + \sqrt{3}) = \frac{8}{27}(-1 + 3i\sqrt{3})$$

$$X_{(-i)2,-1}^{+} = \frac{2}{3\sqrt{3}} + \frac{2i}{3} - \frac{2}{3}(i + \sqrt{3}) = -\frac{4}{3\sqrt{3}}$$

$$X_{(-i)3,0}^{+} = \frac{64}{81\sqrt{3}} + 0 + \frac{32}{27}(i + \sqrt{3}) = \frac{32}{243}(9i + 11\sqrt{3})$$

$$X_{(-i)3,3}^{+} = \frac{10112}{19683} + 0 + \frac{320}{2187}(1 - i\sqrt{3}) = \frac{64}{19683}(203 - 45i\sqrt{3})$$

and

$$X_{(-i)2,0}^{-} = \frac{16}{27} + 0 - \frac{8}{9}(1 + i\sqrt{3}) = -\frac{8}{27}(1 + 3i\sqrt{3})$$

$$X_{(-i)2,-1}^{-} = -\frac{2}{3\sqrt{3}} + \frac{2i}{3} + \frac{2}{3}(-i + \sqrt{3}) = \frac{4}{3\sqrt{3}}$$

$$X_{(-i)3,0}^{-} = -\frac{64}{81\sqrt{3}} + 0 - \frac{32}{27}(-i + \sqrt{3}) = -\frac{32}{243}(-9i + 11\sqrt{3})$$

$$X_{(-i)3,3}^{-} = \frac{10112}{19683} + 0 + \frac{320}{2187}(1 + i\sqrt{3}) = \frac{64}{19683}(203 + 45i\sqrt{3})$$

These results perfectly agree with the ones computed directly using (1.7) (multiplied by $\mathcal{C}$ to get the corresponding X's).

D.3 Reconstruction of $T_{(-i)}^{(N)}$

Analogously to the reconstruction of $T_{(i)}^{(N)}$, we must choose a contour to compute the contribution of the continuous spectrum. In this case, we should choose a contour $C_N^{(-)}$ which stretches from $-\infty$ to ∞ with $\Im(\kappa)$ just below $-\frac{4}{N}$. When we move the contour up toward the real axis, we get two contributions coming from the poles at $\kappa_0 = 0$ (actually, half of it) and $\kappa_1 = -\frac{4i}{N}$, that is

$$
\begin{aligned}
T_{(-i)nm}^{(N)} &= \int_{C_N^{(-)}} d\kappa \; \mu(N,\kappa) V_n^{(2)}(\kappa) V_m^{(-1)}(\kappa) + iP_{nm} \\
&= \int_{-\infty}^{\infty} d\kappa \; \mu(N,\kappa) V_n^{(2)}(\kappa) V_m^{(-1)}(\kappa) + \oint_{\kappa_1} d\kappa \; \mu(N,\kappa) V_n^{(2)}(\kappa) V_m^{(-1)}(\kappa) \\
&\quad + \frac{1}{2} \oint_{\kappa_0} d\kappa \; \mu(N,\kappa) V_n^{(2)}(\kappa) V_m^{(-1)}(\kappa) + iP_{nm}
\end{aligned}
\tag{D.7}
$$

where $\mu(N,\kappa)$ was defined in (7.8). The first term is just $T_{nm}^{(N)}$, and computing the residues we get

$$
T_{(-i)nm}^{(N)} = T_{nm}^{(N)} + \frac{2i}{N} V_n^{(2)}(0) V_m^{(-1)}(0) + \frac{4i}{N} V_n^{(2)}\left(-\frac{4i}{N}\right) V_m^{(-1)}\left(-\frac{4i}{N}\right)
\tag{D.8}
$$

Here are some examples for $N = 4$

$$
\begin{aligned}
T_{(-i)2,-1}^{(4)} &= 0 + \frac{i}{2} + i = \frac{3i}{2} \\
T_{(-i)3,1}^{(4)} &= \frac{11}{16} + 0 + \frac{1}{2} = \frac{19}{16} \\
T_{(-i)4,2}^{(4)} &= \frac{5}{16} + 0 - \frac{5}{4} = -\frac{15}{16} \\
T_{(-i)5,4}^{(4)} &= 0 + 0 - \frac{5i}{16} = -\frac{5i}{16}
\end{aligned}
$$

This agrees precisely with the components of the dual wedge state with Neumann coefficients (one should multiply it by $\mathcal{C}$, since $T_{(-i)}^{(N)} = \mathcal{C} S_{(-i)}^{(N)}$)

$$
S_{(-i)pM}^{(N)} = \oint_0 \frac{dz}{2\pi i} \oint_0 \frac{dw}{2\pi i} \frac{1}{z^{p-1}} \frac{1}{w^{M+2}} \cdot \left[\frac{f_N'(z)^2}{f_N'(w)} \frac{1}{f_N(z) - f_N(w)} - \frac{1}{z-w} \right]
$$

From (D.8) we see that $T_{(-i)}^{(N)} = \mathcal{C} T_{(i)}^{(N)} \mathcal{C}$, as expected.

E A simple proof of $U_{(i)} U_{(-i)} = 1$

From the argument of the path shifting it is obvious that the twisted matrices of the vertices will commute between themselves. But it is instructive to see this directly using our usual $U^2 = 1$ argument.

Let us start by tracing back the U matrices. The relevant Neumann coefficients are defined by

$$\langle \hat{V}_{(i)}| = \langle V_3|Y(i) = \langle \hat{0}|\, e^{-c_n^r V_{nM}^{rs}\, b_M^s} \tag{E.1}$$

$$V_i^{rs}(z,w) = \left[\frac{\partial f_3^r(z)^2}{\partial f_3^s(w)} \frac{1}{f_3^r(z) - f_3^s(w)} \left(\frac{f_3^s(w)}{f_3^r(z)} \right)^3 - \frac{\delta^{rs}}{z-w} \right]$$

$$= \hat{V}^{rs}(z,w) - \frac{4i}{3}\left(\alpha^{r-s} f_{-\frac{4i}{3}}^{(2)}(z) f_{\frac{4i}{3}}^{(-1)}(w) + \frac{1}{2} f_{k=0}^{(2)}(z) f_{k=0}^{(-1)}(w) \right)$$

$$= \int_{\Im\kappa > \frac{4}{3}} \frac{v^{rs}(\kappa)}{2\sinh\frac{\pi\kappa}{2}} f_{-\kappa}^{(2)}(z) f_\kappa^{(-1)}(w), \quad \texttt{Only in the bulk} \tag{E.2}$$

where $\xi_3 = \frac{4i}{3}$.

From now on, for simplicity, we focus on the bulk. This allows to discard the $\kappa = 0$ contribution and concentrate on the rest. The vertex is not twist invariant but still it is cyclic, so we can write (only bulk)

$$V_{(i)}^{rs} = \frac{1}{3}(E_{(i)} + \alpha^{s-r} U_{(i)} + \alpha^{r-s} \bar{U}_{(i)}) \tag{E.3}$$

From the explicit reconstruction formula presented in the main text, we can see that (in the bulk) the only difference wrt the average vertex is in the matrix $\bar{U}_{(i)}$ (the bulk–residual contribution is proportional to α^{r-s}). So, using a subscript "$_0$" for principal parts, we have (we inaugurate a 'bra–ket' notation

$|x\rangle \to V^{(2)}(x)$, $\langle x| \to V^{(-1)}(x)$)

$$E_{(i)} = \mathcal{C}$$
$$U_{(i)} = U_0 = U$$
$$\bar{U}_{(i)} = \bar{U}_0 - \frac{4i}{3}|-\xi_3\rangle\langle\xi_3| \equiv \mathcal{C}U\mathcal{C} - \beta. \tag{E.4}$$

It is easy to see that the commutation of twisted bulk matrices $(\mathcal{C}V_{(i)}^{rs})$ will work iff

$$\mathcal{C}U_{(i)}\mathcal{C}\bar{U}_{(i)} - \mathcal{C}\bar{U}_{(i)}\mathcal{C}U_{(i)} = (\mathcal{C}U\mathcal{C}\bar{U} - \mathcal{C}\bar{U}\mathcal{C}U) - (\mathcal{C}U\mathcal{C}\beta - \mathcal{C}\beta\mathcal{C}U) = 0.$$

Since we know that $U^2 = \bar{U}^2 = 1$, we only have to prove that $\mathcal{C}U\mathcal{C}\beta - \mathcal{C}\beta\mathcal{C}U$. We can actually do better: we can prove that

$$\mathcal{C}U\mathcal{C}\beta = \mathcal{C}\beta\mathcal{C}U = 0, \tag{E.5}$$

which means

$$\mathcal{C}U\,|\xi_3\rangle = 0, \qquad \langle\xi_3|\,\mathcal{C}U = 0 \tag{E.6}$$

It takes a line to compute this quantities using reconstruction formulas. We have indeed

$$\mathcal{C}U = \int_R \frac{d\kappa}{2\sinh\frac{\pi\kappa}{2}}\, u(\kappa)|\kappa\rangle\langle\kappa| \tag{E.7}$$

The $\mathcal{C}U$ eigenvalue can be computed from the known eigenvalues of X^{rs} and it turns out to be

$$u(\kappa) = x^{11}(k) + \bar{\alpha}x^{12}(\kappa) + \alpha x^{21}(\kappa) = 2\left(\cosh\frac{\pi(\kappa - \xi_3)}{2} - 1\right)\frac{\sinh\frac{\pi\kappa}{4}}{\sinh\frac{3\pi\kappa}{4}} \qquad \text{(E.8)}$$

Here we come to the point: u eigenvalues are *vanishing* at $\kappa = \frac{4i}{3}$, while they are divergent at $\kappa = -\frac{4i}{3}$

$$u(\xi_3) = 0, \qquad u(-\xi_3) = \infty.$$

So, in the κ-UHP, there are no poles and the delta functions will work without producing any divergence

$$\mathcal{C}U|\xi_3\rangle = u(\xi_3)|\xi_3\rangle = 0, \qquad \langle\xi_3|\mathcal{C}U = u(\xi_3)\langle\xi_3| = 0 \qquad \text{(E.9)}$$

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